

Foundations of Engineering Mechanics

Ranjan Ganguli

Finite Element Analysis of Rotating Beams

Physics Based Interpolation

 Springer

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Preface

Rotating beams are important mathematical models for structures such as helicopter rotor blades, wind turbine rotor blades, propellers, turbine blades and robotic manipulators. Therefore, the modeling and analysis of rotating beams is an important practical problem. The natural frequencies of the rotating beam should be kept away from multiples of the rotor speed. Therefore, accurate frequency prediction is important. Control of rotating beams requires the development of low-order models. This motivates the development of efficient mathematical models. Since the rotating beam equation does not have a simple exact solution, approximate methods such as Rayleigh–Ritz, Galerkin, and the finite element methods are widely used for the vibration analysis of rotating beams. This book provides an introduction to the finite element for rotating beams. A background on the Rayleigh–Ritz and Galerkin method is also provided.

The first chapter gives a detailed introduction to the rotating beam equation and illustrates the Rayleigh–Ritz, Galerkin, and finite element methods for its solution. Several example problems are given to illustrate the methods. A MATLAB finite element code for the rotating beam is also provided. The following chapters give adaptations of the basis functions which can accelerate the convergence of rotating beam finite elements, thus allowing for efficient low-order models. Some of these basis functions are based on analogies between the piano string and rotating Euler–Bernoulli and between the violin string and rotating Timoshenko beams. A major theme here is to choose finite element interpolation functions which are closer to the problem physics.

This book should be useful to engineers, graduate students and researchers working on rotating beam problems. It is also useful for people in the area of computational mechanics and applied mathematics.

The author is grateful to his advisors Dr. P.K. Datta and Dr. Indrajit Chopra for introducing him to the rotating beam problem. He is also grateful to his students

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Bengaluru, India
2016

Ranjan Ganguli

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About the Author

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Chapter 1

Introduction

1.1 Introduction

Rotating beams are useful mathematical models for helicopter rotor blades, wind turbine blades, turbine blades, robotic manipulators and propeller blades. Therefore, the rotating beam equation is a topic of great practical interest. In this chapter, we derive the equation of motion of the rotating beam and present some approximate methods for its solution. Approximate methods are very useful for obtaining frequencies of rotating beams. The Galerkin method and Rayleigh–Ritz method can be used for this purpose.

1.1.1 Elastic Blade

A good model of a rotor blade is an elastic beam restrained at the root, as shown in Fig. 1.1. Typically, helicopter rotor blades are treated as long slender beams. Treating the blade as a slender beam is appropriate since cross-sectional dimensions are much smaller than the length. At first, we will consider only out-of-plane or flap bending. Also, we make the Bernoulli–Euler assumption: a plane section normal to the neutral axis (beam axis) remains plane after deformation. It is also assumed that shear deflection is negligible and rotary inertia is neglected. This implies that the effect of rotation of element is small compared to vertical displacement. In addition, structural damping is neglected. With these assumptions, we are ready to derive the elastic blade governing differential equation.

Consider a blade section as shown in Fig. 1.2. Here $f_z(r, t)$ is the vertical load per unit length and $w(r, t)$ is the vertical deflection at station r . Several forces act on this blade section: $m\ddot{w}dr$ is the inertia force, $f_z dr$ is the external force, $f_H dr$ is the axial force parallel to r , T is the axial force (Positive for tension), S is the shear force at the cut section (Positive as shown), M is the bending moment (Positive when top fiber is under compression).

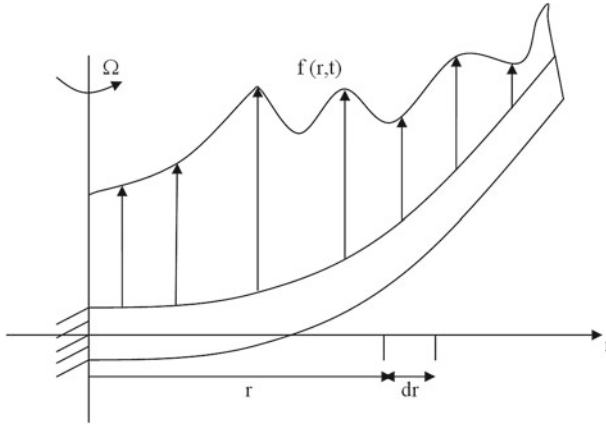
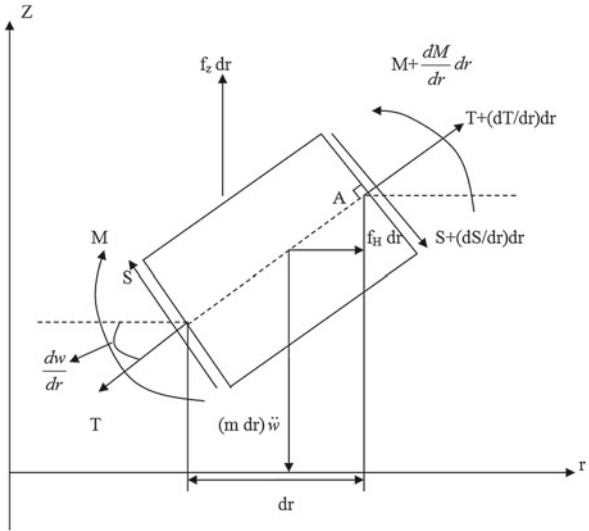


Fig. 1.1 Elastic blade with distributed load

Fig. 1.2 Elastic blade section



Also, $\frac{dw}{dr}$ is the bending deflection slope, assumed to be small. Therefore,

$$\cos\left(\frac{dw}{dr}\right) \approx 1 \quad \sin\left(\frac{dw}{dr}\right) = \frac{dw}{dr} \tag{1.1}$$

Equilibrium of forces on element, in the r-direction yields

$$\sum F_r = -T + T + \frac{dT}{dr}dr + f_H dr - S \frac{dw}{dr} + \left(S + \frac{dS}{dr}dr\right) \left(\frac{dw}{dr} + \frac{d^2w}{dr^2}dr\right) = 0$$

$$\Rightarrow \frac{dT}{dr} + f_H + \frac{d}{dr} \left(S \frac{dw}{dr} \right) = 0 \quad (1.2)$$

Forces in the Z-direction yield,

$$\begin{aligned} \sum F_Z &= f_z dr + S - S - \frac{dS}{dr} dr - m\ddot{w} dr - T \frac{dw}{dr} + \left(T + \frac{dT}{dr} \right) \left(\frac{dw}{dr} + \frac{d^2w}{dr^2} dr \right) = 0 \\ \Rightarrow f_z - \frac{dS}{dr} - m\ddot{w} + T \frac{d^2w}{dr^2} + \frac{dT}{dr} \frac{dw}{dr} &= 0 \end{aligned} \quad (1.3)$$

Or

$$\frac{dS}{dr} = f_z - m\ddot{w} + \frac{d}{dr} \left(T \frac{dw}{dr} \right)$$

The above equation tells that differential of shear is loading distribution.

Now, we take moment about the point A in Fig. 1.2,

$$\begin{aligned} \sum M &= M + Sdr - M - \frac{dM}{dr} dr = 0 \\ \frac{dM}{dr} &= S \end{aligned} \quad (1.4)$$

Thus, differential of moment is shear distribution. From beam theory,

$$M = EI \frac{d^2w}{dr^2} \quad (1.5)$$

Also,

$$S = \frac{dM}{dr} = \frac{d}{dr} EI \frac{d^2w}{dr^2} \quad (1.6)$$

$$\frac{\partial^2}{\partial r^2} \left(EI \frac{\partial^2 w}{\partial r^2} \right) + m \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial r} \left(T \frac{\partial w}{\partial r} \right) = f_z(r, t) \quad (1.7)$$

The above equation is the PDE for beam bending, Force equilibrium in horizontal direction gives,

$$\frac{dT}{dr} + f_H + \frac{d}{dr} \left(S \frac{dw}{dr} \right) = 0 \quad (1.8)$$

$$\frac{d}{dr} \left(S \frac{dw}{dr} \right) \rightarrow \text{Second Order term} \approx 0$$

$$\frac{dT}{dr} + f_H = 0. \quad (1.9)$$

To get a physical feel of the problem, we consider a rectangular cross-section as shown in Fig. 1.3.

$$I = \frac{bh^3}{12}$$

m = mass/unit length and E = Young's Modulus. The unit of I is m^4 , unit of m is Kg/m and the unit of E is GPa.

1.1.2 Horizontal Force Equilibrium

Consider a case with uniform mass and a small mass element from Fig. 1.4. From Eq. (1.9) we get,

$$f_H = m\Omega^2 r \quad (1.10)$$

$$\frac{dT}{dr} = -m\Omega^2 r \quad (1.11)$$

Integrating,

$$T = -m\Omega^2 \frac{r^2}{2} + C \quad (1.12)$$

At tip, $r = R$, $T = 0$, $\Rightarrow C = m\Omega^2 \frac{R^2}{2}$

Fig. 1.3 Rectangular cross-section

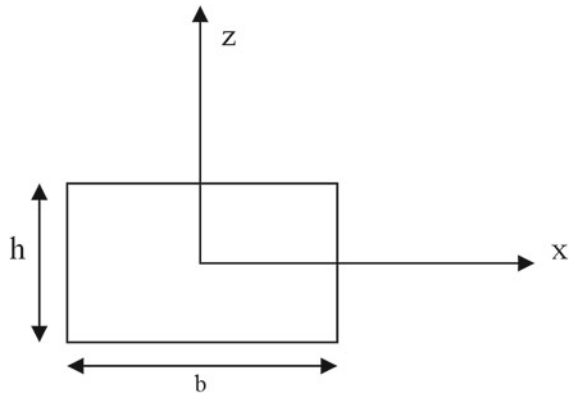
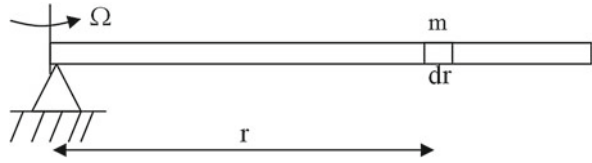


Fig. 1.4 Horizontal force equilibrium



$$T = \frac{m\Omega^2}{2} (R^2 - r^2) \tag{1.13}$$

In general,

$$T = \int_r^R m\Omega^2 r dr = \int_r^R mr\Omega^2 dr \tag{1.14}$$

1.1.3 Boundary Conditions

The governing equation is a fourth order partial differential equation (PDE) in space and needs 4 boundary conditions. These boundary conditions depend on the physics of the problem (Figs. 1.5, 1.6, 1.7, 1.8 and 1.9). We look at some typical boundary conditions.

(a) Pin end or simply supported (Fig. 1.5)

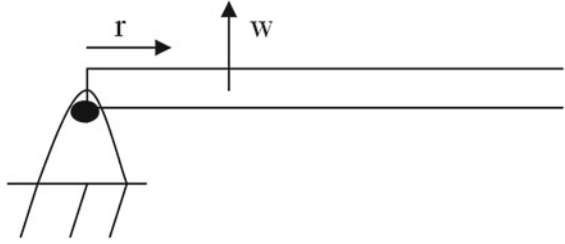
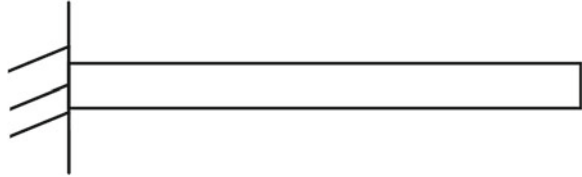
$$w = 0; M = EI \frac{d^2w}{dr^2} = 0$$

(b) Cantilevered or fixed end (Fig. 1.6)

$$w = 0; \frac{dw}{dr} = 0$$

(c) Free end (Fig. 1.7)

$$M = EI \frac{d^2w}{dr^2} = 0; S = \frac{d}{dr} \left(EI \frac{d^2w}{dr^2} \right) = 0$$

Fig. 1.5 Pinned end**Fig. 1.6** Fixed end**Fig. 1.7** Free end

(d) Vertical spring (Fig. 1.8)

$$M = EI \frac{d^2 w}{dr^2} = 0; S = \frac{d}{dr} \left(EI \frac{d^2 w}{dr^2} \right) = -Kw$$

(e) Leaf spring (Fig. 1.9)

$$M = EI \frac{d^2 w}{dr^2} = K_\theta \frac{dw}{dr}; S = \frac{d}{dr} \left(EI \frac{d^2 w}{dr^2} \right) = 0$$

1.1.4 Initial Conditions

The beam PDE is second order in time and so needs two initial conditions: at $t = 0$, $\frac{\partial w}{\partial t}$ and w are prescribed.

1.1.5 Cantilever Beam Vibrations (Non-rotating)

Recall the PDE of the rotating beam,

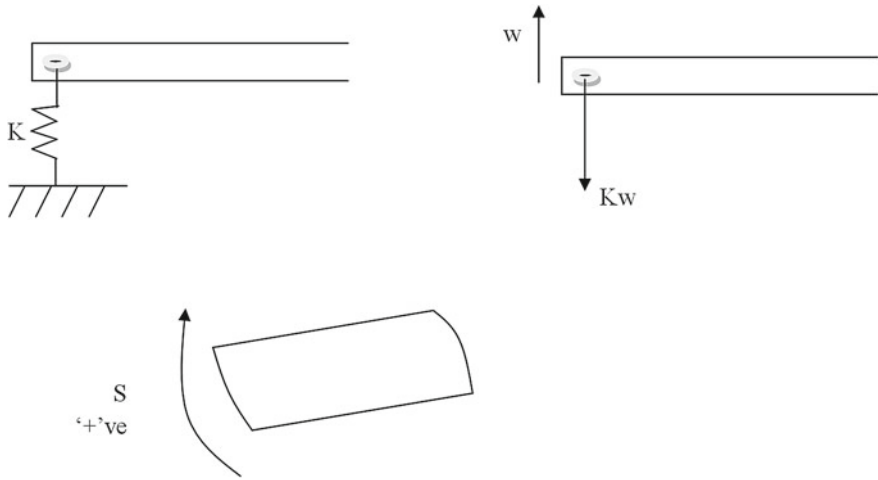


Fig. 1.8 Vertical spring

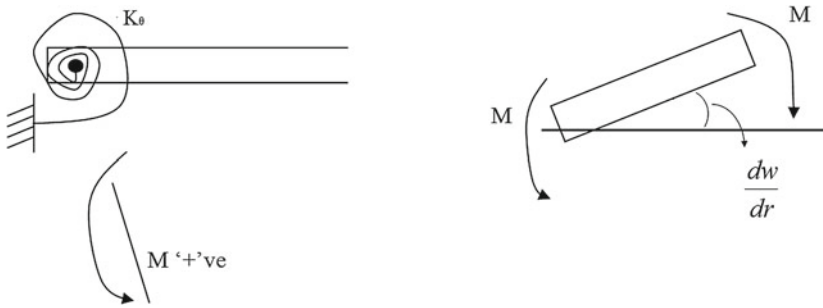


Fig. 1.9 Leaf spring

$$\frac{\partial^2}{\partial r^2} \left(EI \frac{\partial^2 w}{\partial r^2} \right) + m \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial r} \left(T \frac{\partial w}{\partial r} \right) = f_z(r, t) \tag{1.15}$$

For no rotation speed, $T = 0$. If we assume that the mass and stiffness are constant or uniform along the beam,

$$EIw'''' + m\ddot{w} = f_z(r, t) \tag{1.16}$$

If we consider the cantilever boundary conditions:

$$r = 0, w = 0, w' = 0$$

$$r = R, M = EIw'' = 0, w'' = 0; S = EIw''' = 0, w''' = 0$$