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ELEMENTS

OF THE

INTEGRAL CALCULUS,

WITH A

*KEY TO THE SOLUTION OF DIFFERENTIAL
EQUATIONS, AND A SHORT TABLE
OF INTEGRALS.*

BY

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SECOND EDITION, REVISED AND ENLARGED.

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PREFACE.

THE following volume is a sequel to my treatise on the Differential Calculus, and, like that, is written as a text-book. The last chapter, however, a Key to the Solution of Differential Equations, may prove of service to working mathematicians.

I have used freely the works of Bertrand, Benjamin Peirce, Todhunter, and Boole; and I am much indebted to Professor J. M. Peirce for criticisms and suggestions.

I refer constantly to my work on the Differential Calculus as Volume I.; and for the sake of convenience I have added Chapter V. of that book, which treats of Integration, as an appendix to the present volume.

W. E. BYERLY.

CAMBRIDGE, 1881.

PREFACE TO SECOND EDITION.

IN enlarging my Integral Calculus I have used freely Schlömilch's "Compendium der Höheren Analysis," Cayley's "Elliptic Functions," Meyer's "Bestimmte Integrale," Forsyth's "Differential Equations," and Williamson's "Integral Calculus."

The chapter on Theory of Functions was sketched out and in part written by Professor B. O. Peirce, to whom I am greatly indebted for numerous valuable suggestions touching other portions of the book, and who has kindly allowed me to have his Short Table of Integrals bound in with this volume.

W. E. BYERLY.

CAMBRIDGE, 1888.

ANALYTICAL TABLE OF CONTENTS.

CHAPTER I.

SYMBOLS OF OPERATION.

Article.	Page.
1. Functional symbols regarded as <i>symbols of operation</i>	1
2. Compound function; compound operation	1
3. Commutative or relatively free operations	1
4. Distributive or linear operations	2
5. The compounds of <i>distributive</i> operations are distributive	2
6. Symbolic exponents	2
7. The <i>law of indices</i>	2
8. The interpretation of a zero exponent	3
9. The interpretation of a negative exponent	3
10. When operations are commutative and distributive, the symbols which represent them may be combined as if they were algebraic quantities	3

CHAPTER II.

IMAGINARIES.

11. Usual definition of an <i>imaginary</i> . Imaginaries first forced upon our attention in connection with quadratic equations	5
12. Treatment of imaginaries purely arbitrary and conventional	6
13. $\sqrt{-1}$ defined as a symbol of operation	6
14. The rules in accordance with which the symbol $\sqrt{-1}$ is used. $\sqrt{-1}$ <i>distributive</i> and <i>commutative</i> with symbols of quantity	7
15. Interpretation of powers of $\sqrt{-1}$	7
16. Imaginary roots of a quadratic	8
17. Typical form of an imaginary. Equal imaginaries	8
18. Geometrical representation of an imaginary. <i>Reals</i> and <i>pure imaginaries</i> . An interpretation of the operation $\sqrt{-1}$	8
19. The <i>sum</i> , the <i>product</i> , and the <i>quotient</i> of two imaginaries, $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$, are imaginaries of the typical form	10

Article	Page.
20. Second typical form $r(\cos \phi + \sqrt{-1} \sin \phi)$. <i>Modulus and argument. Absolute value of an imaginary. Examples</i>	10
21. The modulus of the sum of two imaginaries is never greater than the sum of their moduli	11
22. Modulus and argument of the product of imaginaries	12
23. Modulus and argument of the quotient of two imaginaries	13
24. Modulus and argument of a power of an imaginary	13
25. Modulus and argument of a root of an imaginary. Example	14
26. Relation between the n n th roots of a real or an imaginary	14
27. The imaginary roots of 1 and -1 . Examples	15
28. <i>Conjugate</i> imaginaries. Examples	17
29. <i>Transcendental</i> functions of an imaginary variable best defined by the aid of series	17
30. Convergency of a series containing imaginary terms	18
31. <i>Exponential</i> functions of an imaginary. Definition of e^z where z is imaginary	19
32. The <i>law of indices</i> holds for imaginary exponentials. Example	20
33. <i>Logarithmic</i> functions of an imaginary. Definition of $\log z$. $\log z$ a <i>periodic</i> function. Example	21
34. <i>Trigonometric</i> functions of an imaginary. Definition of $\sin z$ and $\cos z$. Example	22
35. $\sin z$ and $\cos z$ expressed in <i>exponential</i> form. The fundamental formulas of Trigonometry hold for imaginaries as well as for reals. Examples	22
36. <i>Differentiation</i> of Functions of Imaginary Variables. The <i>derivative</i> of a function of an imaginary is in general indeterminate	24
37. In differentiating, we may treat the $\sqrt{-1}$ like a constant factor. Example. Two forms of the differential of the independent variable	24
38. Differentiation of a power of z	25
39. Differentiation of e^z . Example	26
40. Differentiation of $\log z$	26
41. Differentiation of $\sin z$ and $\cos z$	26
42. Formulas for <i>direct integration</i> (I., Art. 74) hold when x is imaginary	27
43. <i>Hyperbolic Functions</i>	27
44. Examples. Properties of Hyperbolic Functions	28
45. Differentiation of Hyperbolic Functions	28
46. Anti-hyperbolic functions. Examples	28
47. Anti-hyperbolic functions expressed as logarithms	29
48. Formulas for the direct integration of some irrational forms	30

CHAPTER III.

GENERAL METHODS OF INTEGRATING.

Article.	Page.
49. Integral regarded as the inverse of a <i>differential</i>	32
50. If fx is any function whatever of x , $fx.d x$ has an integral, and but one, except for the presence of an arbitrary constant	32
51. A <i>definite integral</i> contains no arbitrary constant, and is a function of the values between which the sum is taken. Examples	33
52. Definite integral of a <i>discontinuous</i> function	33
53. Formulas for direct integration	34
54. Integration by <i>substitution</i> . Examples	36
55. Integration by <i>parts</i> . Examples. Miscellaneous examples in integration	37

CHAPTER IV.

RATIONAL FRACTIONS.

56. Integration of a <i>rational algebraic polynomial</i> . Rational fractions, <i>proper</i> and <i>improper</i>	40
57. Every proper rational fraction can be reduced to a sum of simpler fractions with constant numerators	40
58. Determination of the numerators of the partial fractions by indirect methods. Examples	42
59. Direct determination of the numerators of the partial fractions	43
60. Illustrative examples	45
61. Illustrative example	46
62. Integration of the partial fractions	48
63. Treatment of imaginary values which may occur in the partial fractions. Examples	49

CHAPTER V.

REDUCTION FORMULAS.

64. Formulas for raising or lowering the exponents in the form $x^{m-1}(a + bx^n)^p dx$	52
65. Consideration of special cases. Examples	54

CHAPTER VI.

IRRATIONAL FORMS.

Article.	Page.
66. Integration of the form $f(x, \sqrt[n]{a+bx})dx$. Examples	56
67. Integration of the form $f(x, \sqrt[n]{c+\sqrt[m]{a+bx}})dx$. Examples . .	57
68. Integration of the form $f(x, \sqrt{a+bx+cx^2})dx$	57
69. Illustrative example. Examples	59
70. Integration of the form $f\left(x, \sqrt[n]{\frac{ax+b}{lx+m}}\right)dx$. Example	61
71. Application of the <i>Reduction Formulas</i> of Chapter V. to irrational forms. Examples	61
72. A function rendered irrational through the presence under the radical sign of a polynomial of higher degree than the second cannot ordinarily be integrated. Elliptic Integrals .	62

CHAPTER VII.

TRANSCENDENTAL FUNCTIONS.

73. Use of the method of <i>Integration by Parts</i> . Examples	63
74. Reduction Formulas for $\sin^n x$ and $\cos^n x$. Examples	64
75. Integration of $(\sin^{-1} x)^n dx$. Examples	65
76. Use of the method of <i>Integration by Substitution</i>	66
77. Substitution of $z = \tan \frac{x}{2}$ in integrating trigonometric forms .	67
78. Integration of $\sin^n x \cos^n x dx$. Examples	68
79. Reduction formulas for $\int \tan^n x dx$ and $\int \frac{dx}{\tan^n x}$. Examples . .	69

CHAPTER VIII.

DEFINITE INTEGRALS.

80. Definition of a <i>definite integral</i> as the limit of a sum of infinitesimals	71
81. Computation of a definite integral as the limit of a sum. Illustrative examples. Examples	72
82. Usual method of obtaining the value of a definite integral. Caution concerning multiple-valued functions. Examples .	76

Article.	Page.
83. Consideration of the nature of the value of $\int_a^b fx \cdot dx$ when fx becomes infinite for $x=a$, or $x=b$, or for some value of x between a and b . Illustrative examples	78
84. Test that must be satisfied in order that $\int_a^b fx \cdot dx$ may be finite and determinate if fx is infinite for some value of x between a and b . Illustrative examples. Examples	80
85. Meaning of $\int_a^\infty fx \cdot dx$. Condition that $\int_a^\infty fx \cdot dx$ shall be finite and determinate	84
86. <i>Maximum-Minimum Theorem</i> . Proof that certain important definite integrals of the form $\int_a^\infty fx \cdot dx$ are finite and determinate. Examples	85
87. Application of <i>reduction formulas</i> to definite integrals. Examples	88
88. Application of the method of <i>integration by substitution</i> to definite integrals. Illustrative examples. Example	90
89. Differentiation of a definite integral. Examples	93
90. Many ingenious methods of finding the values of definite integrals are valid only in case the integral is finite and determinate	96
91. Integration by development in series. Examples	96
92. Values of $\int_0^{\frac{\pi}{2}} \log \sin x \cdot dx$, $\int_0^\infty e^{-x^2} dx$, $\int_0^\infty \frac{\sin mx}{x} \cdot dx$ obtained by ingenious devices. Examples	98
93. Differentiation or integration with respect to a quantity which is independent of x . Examples	101
94. Additional illustrative examples. Examples	102
95. Introduction of imaginary constants.	104
96. The <i>Gamma Function</i>	105
97. Table giving $\log \Gamma(n)$ from $n=1$ to $n=2$. Definite integrals expressed as Gamma Functions	107
98. The <i>Beta Function</i> . Formula connecting the Beta Function with the Gamma Function. Value of $\Gamma(\frac{1}{2})$	109
99. More definite integrals expressed as Gamma Functions. Examples	110

CHAPTER IX.

LENGTHS OF CURVES.

Article.	Page.
100. Formulas for $\sin \tau$ and $\cos \tau$ in terms of the length of the arc	113
101. The equation of the <i>Catenary</i> obtained. Example	113
102. The equation of the <i>Tractrix</i> . Examples	115
103. Length of an arc in rectangular coördinates	117
104. Length of the arc of the <i>Cycloid</i> . Example	118
105. Another method of rectifying the <i>Cycloid</i>	119
106. Rectification of the <i>Epicycloid</i> . Examples	119
107. Arc of the <i>Ellipse</i> . Auxiliary angle. Example	120
108. Length of an arc. Polar coördinates	121
109. Equation of the <i>Logarithmic Spiral</i>	121
110. Rectification of the Logarithmic Spiral. Examples	122
111. Rectification of the <i>Cardioide</i>	122
112. <i>Involutes</i> . Illustrative example. Example	123
113. The <i>involute</i> of the <i>Cycloid</i> . Example	125
114. <i>Intrinsic equation</i> of a curve. Example	126
115. Intrinsic equation of the <i>Epicycloid</i> . Example	127
116. Intrinsic equation of the <i>Logarithmic Spiral</i>	128
117. Method of obtaining the <i>intrinsic equation</i> from the equation in rectangular coördinates. Examples	128
118. Intrinsic equation of an <i>evolute</i>	130
119. Illustrative examples. Examples	130
120. The <i>evolute</i> of an <i>Epicycloid</i> . Example	131
121. The <i>intrinsic equation</i> of an <i>involute</i> . Illustrative examples	132
122. Limiting form approached by an <i>involute</i> of an <i>involute</i> . .	133
123. Method of obtaining the equation in rectangular coördinates from the <i>intrinsic equation</i> . Illustrative example	134
124. Rectification of <i>Curves in Space</i> . Examples	135

CHAPTER X.

AREAS.

125. <i>Areas</i> expressed as definite integrals, rectangular coördinates. Examples	137
126. <i>Areas</i> expressed as definite integrals, polar coördinates . .	139
127. Area between the <i>catenary</i> and the axis	139
128. Area between the <i>tractrix</i> and the axis. Example	139

TABLE OF CONTENTS.

Article.	Page.
129. Area between a curve and its asymptote. Examples . . .	140
130. Area of circle obtained by the aid of an auxiliary angle. Ex- amples	141
131. Area between two curves (rect. coör.). Examples	142
132. Areas in Polar Coördinates. Examples	143
133. Problems in areas can often be simplified by transformation of coördinates. Examples	146
134. Area between a curve and its evolute. Examples	146
135. Holditch's Theorem. Examples	147
136. Areas by a <i>double integration</i> (rect. coör.)	149
137. Illustrative examples. Examples	150
138. Areas by a <i>double integration</i> (polar coör.). Example . . .	151

CHAPTER XI.

AREAS OF SURFACES.

139. Area of a <i>surface of revolution</i> (rect. coör.). Example . . .	153
140. Illustrative examples. Examples	154
141. Area of a surface of revolution by transformation of coördi- nates. Example	155
142. Area of a <i>surface of revolution</i> (polar coör.). Examples . .	157
143. Area of a <i>cylindrical surface</i> . Examples	157
144. Area of <i>any surface</i> by a double integration	160
145. Illustrative example. Examples	163
146. Illustrative example requiring transformation to polar coördi- nates. Examples	165

CHAPTER XII.

VOLUMES.

147. Volume by a single integration. Example	168
148. Volume of a <i>conoid</i> . Examples	169
149. Volume of an <i>ellipsoid</i> . Examples	170
150. Volume of a <i>solid of revolution</i> . Single integration. Exam- ples	171
151. Volume of a <i>solid of revolution</i> . Double integration. Exam- ples	172
152. Volume of a <i>solid of revolution</i> . Polar formula. Example .	174
153. Volume of <i>any solid</i> . Triple integration. Rectangular coör- dinates. Examples	175
154. Volume of <i>any solid</i> . Triple integration. Polar coördinates. Examples	179

CHAPTER XIII.

CENTRES OF GRAVITY.

Article.	Page.
155. Centre of Gravity defined	180
156. General formulas for the coördinates of the Centre of Gravity of any mass. Example	180
157. Centre of Gravity of a homogeneous body	182
158. Centre of Gravity of a <i>plane area</i> . Examples	182
159. Centre of Gravity of a <i>homogeneous solid</i> of revolution. Ex- amples	185
160. Centre of Gravity of an <i>arc</i> ; of a <i>surface of revolution</i> . Ex- amples	187
161. <i>Properties of Guldin</i> . Examples	188

CHAPTER XIV.

LINE, SURFACE, AND SPACE INTEGRALS.

162. <i>Point function</i> . <i>Continuity</i> of a point function	190
163. <i>Line-integral, surface-integral, and space-integral</i> of a point function	190
164. Value of a line, surface, or space integral independent of the position in each element of the point at which the value of the function is taken	191
165. Value of a line, surface, or space integral independent of the manner in which the line, surface, or space is broken up into infinitesimal elements	191
166. Geometrical representation of a line-integral along a plane curve, and a surface-integral over a plane surface	193
167. <i>Moments of inertia</i> . Examples	193
168. Relation between a surface-integral over a plane surface, and a line-integral along the curve bounding the surface. Ex- ample	196
169. Illustrative example. Examples	198
170. Another form of the relation established in Art. 168. Example	199
171. Relation between a space-integral taken throughout a given space and a surface-integral over the surface bounding the space. Example	199
172. Illustrative example. Example	201

CHAPTER XV.

MEAN VALUE AND PROBABILITY.

Article.	Page.
173. References	202
174. <i>Mean value of a continuously varying quantity.</i> The <i>mean distance</i> of all the points of the circumferences of a circle from a fixed point on the circumference. The <i>mean distance</i> of points on the surface of a circle from a fixed point on the circumference. The <i>mean distance</i> of points on the surface of a square from a corner of the square. The <i>mean distance</i> between two points within a given circle	202
175. Problems in the application of the Integral Calculus to <i>probabilities</i> . Random straight lines. Examples	204

CHAPTER XVI.

ELLIPTIC INTEGRALS.

176. Motion of a <i>simple pendulum</i> . <i>Vibration</i> . <i>Complete revolution</i>	211
177. The <i>length</i> of an <i>arc</i> of an <i>Ellipse</i>	213
178. Algebraic forms of the <i>Elliptic Integrals</i> of the first, second, and third class. <i>Modulus</i> . <i>Parameter</i>	213
179. Trigonometric forms of the <i>Elliptic Integrals</i> . <i>Amplitude</i> . <i>Delta</i> . <i>Complementary Modulus</i>	214
180. <i>Landen's Transformation</i> . Reduction formula by which we can increase the modulus and diminish the amplitude of an <i>Elliptic Integral</i> of the first class. A method of computing $F(k, \phi)$	215
181. Reduction formula for diminishing the modulus and increasing the amplitude of an <i>Elliptic Integral</i> of the first class. A second method of computing $F(k, \phi)$	218
182. Actual computation of $F\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $F\left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$	219
183. <i>Landen's Transformation</i> . Reduction formula by which we can increase the modulus and diminish the amplitude of an <i>Elliptic Integral</i> of the second class. A method of computing $E(k, \phi)$	222
184. A reduction formula for diminishing the modulus and increasing the amplitude of an <i>Elliptic Integral</i> of the second class. A second method of computing $E(k, \phi)$	225

Article.	Page
185. Actual computation of $E\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $E\left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$	227
186. An Elliptic Integral of the first or second class, whose amplitude is greater than $\frac{\pi}{2}$, can be made to depend upon one whose amplitude is less than $\frac{\pi}{2}$, and upon the corresponding complete Elliptic Integral	231
187. Three-place table of Elliptic Integrals of the first class and the second class	233
188. <i>Addition Formulas.</i> Functions defined by the aid of definite integrals. $\log x$, $\sin^{-1}x$, $\tan^{-1}x$, $F(k, x)$, $E(k, x)$. Addition formula for $\log x$	235
189. Addition formulas for $\sin^{-1}x$ and $\tan^{-1}x$	237
190. Addition formula for $F(k, x)$	239
191. Analogy between $\log^{-1}u$, $\sin u$, $\tan u$, and $F^{-1}(k, u)$	241
192. The <i>Elliptic Functions</i> , $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$. Their analogy with Trigonometric Functions. Formulas connecting the Elliptic Functions of a single quantity	242
193. Formulas for Elliptic Functions of $(u+v)$ and $(u-v)$	244
194. Formulas for Elliptic Functions of $2u$	246
195. Formulas for Elliptic Functions of $\frac{u}{2}$	246
196. <i>Periodicity</i> of the Elliptic Functions. Real period, $4K$	247
197. Elliptic Functions of a pure imaginary. <i>Jacobi's Transformation.</i> Elliptic Functions have an imaginary period, $4K'\sqrt{-1}$. Table of values of Elliptic Functions having the modulus $\frac{\sqrt{2}}{2}$	249
198. The Elliptic Integral of the second class expressed in terms of Elliptic Functions. <i>Addition formula</i> for Elliptic Integrals of the second class	252
199. <i>Application to the rectification of the Lemniscate.</i> Examples. Bisection of the arc of a quadrant of the Lemniscate.	254
200. Rectification of the Ellipse. Examples	257
201. Use of the Addition Formula in dealing with Elliptic arcs. <i>Fagnani's Point.</i> Examples	258
202. Rectification of the <i>Hyperbola.</i> Examples	260
203. The <i>simple pendulum.</i> Examples	263

CHAPTER XVII.

INTRODUCTION TO THE THEORY OF FUNCTIONS.

Article.	Page.
204. <i>Single-valued</i> functions. <i>Multiple-valued</i> functions	267
205. Importance of the graphical representation of imaginaries. <i>Complex quantity</i>	267
206. When a complex variable is said to vary continuously	268
207. A <i>continuous</i> function of a complex variable. <i>Critical values</i>	268
208. Criterion that a function shall have a determinate derivative. <i>Monogenic</i> functions	269
209. Any function involving z as a whole is a monogenic function	272
210. <i>Conjugate</i> functions. Their use as solutions of Laplace's Equation. Example	272
211. <i>Preservation of angles</i>	274
212. If two paths traced by the point representing the variable have a common beginning and a common end, and do not enclose a <i>critical point</i> , the corresponding paths traced by the point representing the function and having a common beginning will have a common end	276
213. Examples where the paths traced by the point representing the variable enclose a critical point	278
214. Critical points at which the derivative of the function is zero or infinite are to be avoided. <i>Branch points. Holomorphic</i> functions	279
215. Definite integral of a function of a complex variable defined. Such an integral is generally indeterminate, and depends upon the path by which the point representing the variable passes from the lower limit to the upper limit of the integral	281
216. If the function is <i>holomorphic</i> , the definite integral is in gen- eral determinate	283
217. The integral around a closed contour, embracing a point at which the function is infinite	284
218. Illustrative examples	285
219. Convergency of the series obtained by integrating the terms of a convergent series where the separate terms are holo- morphic functions	287
220. Proof of Taylor's and Maclaurin's Theorems for functions of complex variables. <i>Circle of convergence</i>	288
221. Investigation of the convergency of various series which are obtained by Taylor's and Maclaurin's Theorems. Examples	291

CHAPTER XVIII.

KEY TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

Article.	Page.
222. Description of <i>Key</i>	296
223. Definition of the terms <i>differential equation, order, degree, linear, general solution or complete primitive, singular solution, exact differential equation</i>	296
224. Examples illustrating the use of the <i>Key</i>	298
225. Simplification of differential equations by change of variable.	308
KEY	310
EXAMPLES UNDER KEY	333

INTEGRAL CALCULUS.

CHAPTER I.

SYMBOLS OF OPERATION.

1. It is often convenient to regard a functional symbol as indicating *an operation to be performed upon the expression which is written after the symbol*. From this point of view the symbol is called a *symbol of operation*, and the expression written after the symbol is called the *subject* of the operation.

Thus the symbol D_x in $D_x(x^2y)$ indicates that the *operation* of differentiating with respect to x is to be performed upon the *subject* (x^2y) .

2. If the *result* of one operation is taken as the *subject* of a second, there is formed what is called a *compound function*.

Thus $\log \sin x$ is a *compound function*, and we may speak of the taking of the $\log \sin$ as a *compound operation*.

3. When two operations are so related that the compound operation, in which the result of performing the first on any subject is taken as the subject of the second, leads to the same result as the compound operation, in which the result of performing the second on the same subject is taken as the subject of the first, the two operations are *commutative* or *relatively free*.

Or to formulate; if

$$fFu = Ffu,$$

the operations indicated by f and F are *commutative*.