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Alexandre Almeida
Luís Castro
Frank-Olme Speck
Editors

Advances in Harmonic Analysis and Operator Theory

The Stefan Samko Anniversary Volume

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Editors
Alexandre Almeida
Luís Castro
Departamento de Matemática
Universidade de Aveiro
Aveiro, Portugal

Frank-Olme Speck
Instituto Superior Tecnico
Departamento Matematica
Universidade Tecnica Lisboa
Lisboa, Portugal

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Preface

Harmonic Analysis and Applications is one of the most rapidly developing areas of mathematics at the beginning of the twenty-first century. It has plenty connections to Operator Theory, Functional Analysis, modern Potential Theory, Partial Differential Equations, Boundary Value Problems, Integral and Pseudodifferential Equations, Complex Analysis and, last not least the Theory of Function Spaces. This intersects with the areas of Fractional Calculus and the most recent Variable Exponent Analysis – in $L^{p(\cdot)}$ spaces.

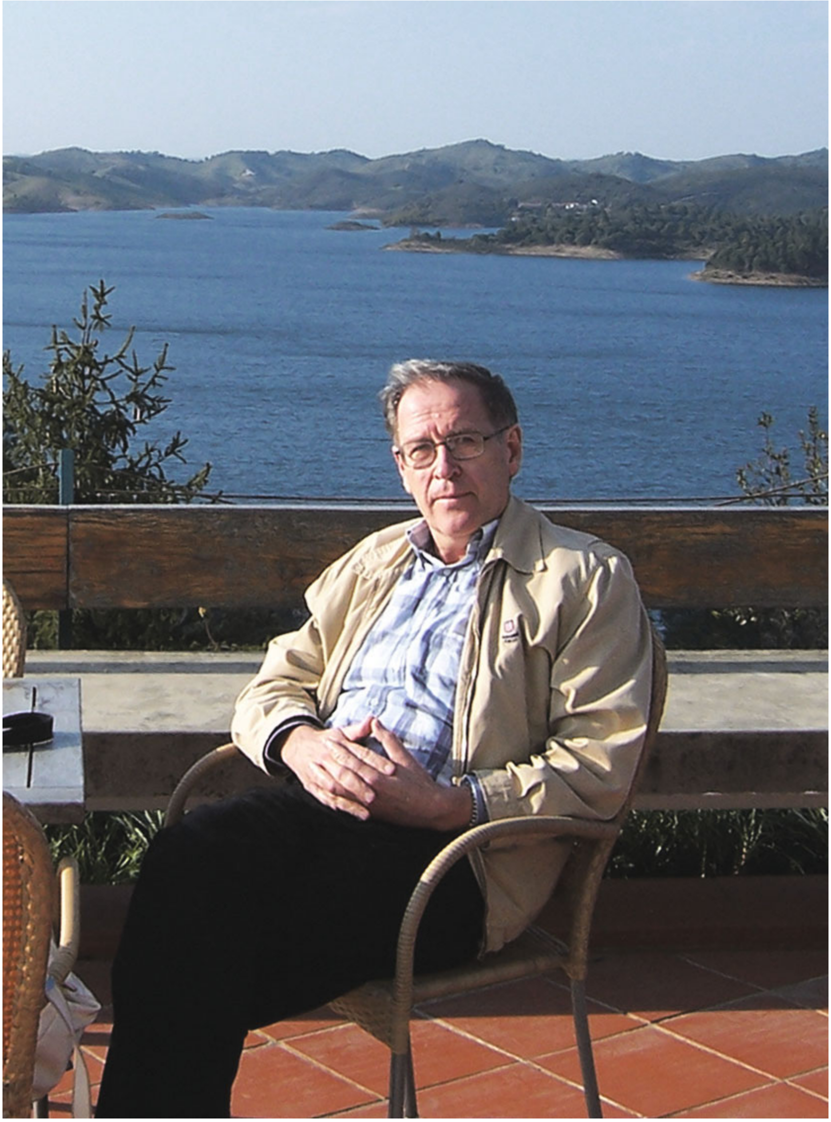
Stefan G. Samko is an exponent of these areas and their developments. His impressive lifework becomes evident in the survey article of Vakhtang Kokilashvili which leads the sequence of articles of this book. The paper of Sergei Rogosin treats some of Stefan Samko's partial interests in a rather compact form. Other contributions of this volume reflect the wide-range and cross connections between the areas mentioned before. Thus, these peer-reviewed research and survey papers will be of interest of a wide-range of readership in pure and applied mathematics.

The seventieth birthday of Stefan Samko on March 28, 2011, has been commemorated in a series of international conferences in Russia and Portugal, here particularly during the IDOTA 2011 on Integral and Differential Operators and Their Applications, held at Aveiro, June 30–July 2, organized by the *Center for Research and Development in Mathematics and Applications*, University of Aveiro. The contributions of this volume are related to lectures of this meeting and a summer school and workshop, STOP 2011 on Selected Topics of Operator Theory, held at Lisbon, June 24–30, which was organized by the *Center for Functional Analysis and Applications*, the research center in Portugal that hosted Stefan Samko within the last fifteen years. These last two scientific events beneficiated from the financial support of Portuguese Science and Technology Foundation ("FCT – Fundação para a Ciência e a Tecnologia") – which we also would like to acknowledge in here.

The editors like to express their gratitude to Stefan Samko for a fruitful and pleasant cooperation within that period of time. We feel happy to honour our hard-working colleague and humorous friend by the edition of this volume, and wish him good health and ever greater success in his work.

Aveiro and Lisbon, in March 2012

Alexandre Almeida, Luís Castro and Frank-Olme Speck



Stefan G. Samko

Operator Theory:
Advances and Applications, Vol. 229, 1–48

Stefan G. Samko – Mathematician, Teacher and Man

V. Kokilashvili

1. Introduction

It is a great honour for me, to contribute this article on the occasion of a remarkable anniversary. The goal of this paper is to present, in a concise manner, the scientific achievements of a leading authority in mathematical analysis, Professor Stefan Samko.

I will mention the following trends of his research.

2. Scientific origin from BVP and SIE, 1965–1974

Stefan Grigorievich Samko started his first research in 1964 in the scientific school of Prof. F.D. Gakhov in the area of Boundary Value Problems (BVP) and Singular Integral Equations (SIE), who was known by the solution of the Riemann boundary value problem. As a graduate student of Rostov State University he defended his diploma (equivalent to Master Degree) under the supervisorship of Prof. Yu. Cherski, it was related to the so-called exceptional cases of an abstract singular integral equation $(A_1 + A_2S)\varphi = f$ in a Banach space, where S satisfies the property $S^2 = I$. Such an equation studied by Cherskii when the operators $A_1 \pm A_2$ were invertible, was now investigated under the assumption that they are not invertible, this study being published in [123], 1965.

One of the main topics of studies in S. Samko's PhD theses was the investigation of solvability of the so-called generalized Abel integral equation

$$u(x)\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\mu} + v(x)\int_x^b \frac{\varphi(t)\,dt}{(t-x)^\mu} = f(x), \quad a < x < b, \tag{2.1}$$

used in particular in mixed boundary value problems. It involves the well-known expressions of the left- and right-hand sided forms of the fractional integration. He found various explicit relations [125], 1967, connecting such forms with each other via singular integral operators. One of them is given in the sequel in (3.1). Relations of such a type allowed to easily reduce equations of form (2.1) and others to weighted singular integral equations. However, in this way there arose a problem of a precise characterization of the space of functions representable as fractional integrals of function in this or other weighted space used in the theory of singular integral equations. Several of his publications were devoted to the solution of this problem together with the study of the solvability of the equations (2.1) and more general integral equations of the first kind in the corresponding spaces of functions, see [124], [127], 1967; [128], 1968; [129], [131], 1969; [133], [134], 1970; [135], [136], [137], 1971; [140], 1975. Some of these results were later summarized in his books [197], 1987, and [198], 1993. The paper [126], 1967, on the reduction of certain integral equations of the first kind arising in elasticity and hydrodynamics to equations of the second kind, is also close to this topic. A detailed study of integral equations of the first kind with a logarithmic type kernel was undertaken in [143], 1976; [154], 1978, where such equations were reduced to singular integral equations jointly with a certain condition of orthogonality to delta-type functions supported at the end points of the interval, which allowed to give a complete picture of solvability of such equations of the first kind.

Another point of the interest, related to his PhD studies was a search of forms of singular integral equations, different from the characteristic singular equations, which admit a solution in closed form. Results of this kind may be found in [130], [132], 1969; [139], 1974.

A brief overview on integral equations of the first kind related to the Riemann boundary problem (or equivalently to singular integral equations), together with an overview of some multidimensional integral equations of the first kind, studied afterwards, was later presented in [162], 1981.

Two papers [15], 1973, and [16], 1975, of that period, inspired by applications, were related to the generalized argument principle for analytic functions vanishing at the boundary.

In this period there appeared a short note [141], 1975, where S. Samko observed that the Babenko-Stein theorem on the boundedness of singular integral operators in L^p with power weight is an immediate consequence of the non-weighted case and the known boundedness of integral operators with kernels homogeneous of order $-n$ of Hardy-Littlewood type.

Two later papers [217], 1984, and [218], 1985, were also related to his interests in one-dimensional singular integrals. In [217] there was studied the classical Hilbert operator

$$Sf(x) = \frac{1}{\pi}\int_a^b \frac{f(t)\,dt}{t-x}$$

in the weighted generalized Hölder space $H_0^\omega([a,b];\varrho)$ with a power weight $\varrho(x) = (x-a)^\alpha(b-x)^\beta$ and obtained a weighted Zygmund type estimate for the continuity modulus of Hf :

$$\omega(\varrho Sf,h) \le c\int\limits_0^h \frac{\omega(\varrho f,t)}{t^\gamma}\,dt + ch\int\limits_h^{b-a}\frac{(\varrho f,t)}{t(t+h)}\,dt$$

in the case $1 \le \alpha < 2, 1 \le \beta < 2$ (the easier case $0 < \alpha < 1, 0 < \beta < 1$ was known earlier), which was applied to obtain the boundedness conditions of the operator S in $H_0^\omega([a,b];\varrho)$. In [218] there was proved a Zygmund type estimate of a conjugate function (the singular operator along a circumference) without weight, but in a more fine setting of moduli of continuity of fractional order of type, see (3.2) in the next section. Similar estimates for hypersingular integrals (HSI) were given in [219], 1986.

3. Research in Fractional Calculus (FC), 1967–1996

FC is a topic which in fact was always of permanent interest of S. Samko. In this section we touch the period of 1967–1996. His studies in this topic after 1996, when he was already in Portugal, are outlined in Section 6.1.

3.1. One-dimensional Fractional Calculus

3.1.1. Relations between left- and right-hand sided fractional integration. The first studies in the area of boundary value problems, singular integral equations and integral equations of the first kind, led S. Samko to a general interest to the Fractional Calculus.

In the sequel S. Samko's interest in this topic covered wide areas in FC, starting from some properties of classical fractional one-dimensional Riemann-Liouville derivatives to multidimensional fractional integration and differentiation, fractional powers of operators, fractional Sobolev type spaces and others. Some of his one-dimensional results in FC were related to the study of equations of the type (2.1). In particular, when studying the equation (2.1), he arrived at various types of operator relations between the operator of the left-hand sided Riemann-Liouville fractional integration and that of the right-hand sided one. One of them has the form

$$\begin{aligned}\int\limits_x^b \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}} = \int\limits_a^x \frac{\cos(\alpha\pi)\varphi(t)\,dt}{(x-t)^{1-\alpha}}& \\ + \frac{1}{\pi}\int\limits_a^x \frac{\sin(\alpha\pi)\,dt}{(t-a)^\alpha(x-t)^{1-\alpha}}\int\limits_a^b \frac{(s-a)^\alpha\varphi(s)\,ds}{s-t}&,\end{aligned} \tag{3.1}$$

being one of the first results in this area. We refer to [125], 1967; [127], 1968; [129], 1969; [133], 1970, for this and other relations of such a type.

3.1.2. Estimates of moduli of continuity. Let

$$\omega_\gamma(f,h) := \sup_{|t|<h} \|\Delta_t^\gamma f\|_X, \quad \gamma > 0, \tag{3.2}$$

be the modulus of continuity of fractional order, where $\Delta_t^\gamma f = (I-\tau_t)^\gamma f,\ \tau_h f(x) = f(x-h)$, and $X = L_p, 1 \le p < \infty$ or $X = C$. It was used in the study of singular integrals in [218], 1985, and HSI in [219], 1986.

We mention the weighted estimate

$$\omega\left(\varrho I_{a+}^\alpha f, h\right) \le ch^{\alpha+\nu-1} \int_0^h \frac{\omega(\varrho f,t)}{t^\nu}\,dt + ch \int_h^{b-a} \frac{\omega(\varrho f,t)}{t^{2-\alpha}}\,dt$$

for the continuity modulus of the Riemann-Liouville fractional integral $I_{a+}^\alpha f(x)$, where $0<\alpha<1, \varrho(x) = (x-a)^\mu,\ 0 \le \mu < 2-\alpha, \quad \nu = \max(1,\mu)$ and $\omega(f,t)$ is the usual continuity modulus (the choice $\gamma = 1$ and $X = C$ in (3.2)), which was obtained in [200], 1987 (the proof in the non-weighted case $\mu = 0$ may be found in the book [198], p. 249; in this case the above estimate includes only the first term). This estimate, together with a similar estimate for fractional derivatives, allowed in [200] to obtain the statement that the Riemann-Liouville-fractional integration operator I_{a+}^α maps the weighted generalized Hölder space $H_0^\omega([a,b],\varrho)$ with such a weight, exactly onto the space $H_0^{\omega_\alpha}([a,b],\varrho)$ with the same weight and better dominant $\omega_\alpha(h) := h^\alpha\omega(h)$ of the continuity moduli, under the assumption that ω belongs to a certain Bari-Stechkin type class.

An extension of such weighted results in the spaces $H_0^\omega([a,b],\varrho)$ to the operators of the form of $\int_0^x k(x-t)f(t)\,dt$ with more general kernels may be found in [201], 1993.

3.1.3. In collaboration with Bertram Ross. In 1992–1993 S. Samko was a Fulbright Professor in the USA, at the University of New Haven, where he obtained several essential results in collaboration with Prof. Bertram Ross, known by his studies in FC and famous as the organizer of the first Conference on FC held in USA in 1974.

In their first paper [202], 1993, they introduced fractional integrals $I_{a+}^{\alpha(x)}$ and derivatives $D_{a+}^{\alpha(x)}$ of Riemann-Liouville type with variable order $\alpha(x)$ (see also [169]), posed some open questions and studied the compositions $I_{a+}^{\alpha(x)} I_{a+}^{\beta(x)}$ with special calculations for the particular case where $\alpha(x)$ is a step function. A Fourier transforms approach via the multiplier $(-i\xi)^{-\alpha(\xi)}$ was also suggested there and conditions on $\alpha(\xi)$ given, which guarantee the existence of a locally integrable kernel of the corresponding convolution operator. This paper in fact was a start of his general interest to what is now called Variable Exponent Analysis, see Section 6.

The papers [116], [170], 1995, are also in the same spirit. In the former, there were studied the mapping properties of the operator $I_{a+}^{\alpha(x)}$ in the Hölder spaces

$$H^{\lambda(\cdot)}([a,b]) := \{f : |f(x+h) - f(x)| \le ch^{\alpha(x)} \le Ch^{\alpha(x)}\}$$

also of variable order; already here the log-condition on $\lambda(x)$ appeared under which it was shown that $I_{a+}^{\alpha(\cdot)}$ maps $H_0^{\lambda(\cdot)}$ into $H_0^{\lambda(\cdot)+\alpha(\cdot)}$, $\sup_x[\lambda(x)+\alpha(x)]<1$. In the latter there were given conditions on $\alpha(x)$ and $\beta(x)$, under which the difference $I_{a+}^{\alpha(\cdot)}I_{a+}^{\beta(\cdot)} - I_{a+}^{\alpha(\cdot)+\beta(\cdot)}$ with $a=-\infty$ is a compact operator in $L^p(\mathbb{R})$.

Another study in FC worth of mentioning, was made in the paper [117], 1994, made at the same period, was related to a question of existence of nowhere differentiable functions which have fractional derivatives. For any $\nu_0>0$ which may be integer in particular, there was constructed a Weierstrass type function $W_{\nu_0}(x)$ which *nowhere has derivative of the order* ν_0, *but has fractional derivatives of every order* $0<\nu<\nu_0$ (*which even satisfy the Hölder condition of order* $\nu_0-\nu$).

3.1.4. Other. As is known, the Liouville forms of fractional integro-differentiation (the left-hand side and right-hand side ones) for functions defined on the whole real line, do not admit functions growing either at $-\infty$ or $+\infty$. In [215], 1992, for Chen's non-convolution type modification I_c^α of the operators of fractional integro-differentiation, related to a given point $c\in\mathbb{R}$ and applicable to functions with an arbitrary growth at $\pm\infty$, there was constructed the corresponding Marchaud form of fractional differentiation $\mathbb{D}_c^\alpha$ and shown that it generates a real left inverse to Chen fractional integrals. This result was developed in [216], 2001, where one can also find a version of non-convolution type identity approximation with L^p-convergence globally or locally, under natural assumptions.

In [19], 1997, S. Samko studied the possibility of influence of the weight on the asymptotics of singular values of the Riemann-Liouville fractional integration operator I_0^α in L^2-spaces and shown that the presence of the weight may cause an asymptotic behaviour different from what might be expected from the smoothness of the kernel of the operator, in the case of both finite or infinite interval. A similar question was also touched for the multidimensional case of the Riesz potential operator.

We also mention his result on the coincidence of the domains of Liouville and Grunwald-Letnikov fractional differentiation operators in [161], 1885; [164], 1990, see also its formulation below for the multidimensional case.

3.2. Multidimensional FC

References to the studies in multidimensional FC are dispersed in the next sections, mainly given in Section 5. Here we single out only the following multidimensional statement with directional multidimensional derivatives, but in one-dimensional direction, obtained in [163], 1990. By X and Y here is denoted one of the spaces $X=L^p(\mathbb{R}^n), 1<p<\infty$ or $X=C?(\dot{\mathbb{R}}^n), Y=L^r(\mathbb{R}^n), 1<r<\infty$, or $Y=C(\dot{\mathbb{R}}^n)$ with $\dot{\mathbb{R}}^n$ denoting the compactification of $\mathbb{R}^n$ by the unique infinite point.

Let $f\in Y$. *The Grünwald-Letnikov derivative*

$$\lim_{\substack{|h|\to 0\\ (X)}} \frac{(\Delta_h^\alpha f)(x)}{|h|^\alpha}$$

and the directional Liouville fractional derivative

$$\lim_{\substack{\varepsilon\to 0\\(X)}} \frac{1}{\varkappa(\alpha,\ell)} \int_{\varepsilon}^{\infty} \frac{(\Delta^{\alpha}_{th'} f)(x)}{t^{1+\alpha}}\, dt, \quad \ell > \alpha,$$

in the direction $h' = \frac{h}{|h|}$ *exist simultaneously and coincide with each other.*

Another multidimensional result in FC we mention here is the realization of the fractional powers $(-|x|^2\Delta)^{\frac{\alpha}{2}}$ of the operator $-|x|^2\Delta$ invariant with respect to dilations which was suggested in [1], 1996, and given with proofs in [2], 2000.

Stefan Samko's research in FC made him known as one of the top experts worldwide known in FC, especially after the publication of the book [197], 1987 (Russian edition) and [198], 1993 (English edition), jointly with A. Kilbas and O. Marichev. He was Chairman of the International Programme Committee of the 1st IFAC Workshop on Fractional Differentiation and its Applications (FDA-04), Bordeaux, France, July 19–21, 2004, and Honorable Chairman of the International Programme Committee of the 2nd IFAC Workshop on Fractional Differentiation and its Applications, Porto, Portugal, 19–21 July, 2006.

4. Equations with involutive operators, 1970–1977

This research was made in collaboration with Nikolai Karapetiants, S. Samko's colleague and close friend. It started in 1969, being the most intensive in 1970–1977 and afterwards continued with intervals till the untimely death of Nikolai in 2005. In this section we overview their results obtained in 1970–1977. In Subsection 6.2 the reader may find the results of the later period 1997–2002.

They started with some problems in the theory of convolution type integral equations, a topic popular in 70s. In particular, they weakened conditions on a function $a(t), t \in \mathbb{R}$ ensuring the compactness of the convolution operator $ak * f$ or $k * (af)$ in the space $L^p(\mathbb{R}), 1 \le p \le \infty$ (as known, the condition $a(x) \to 0$ as $x \to \infty$ is sufficient for that). They showed [27], 1970, that this tendency to zero may hold in a very weak sense, expressed in terms of measure of the set where $a(x) \neq 0$. It is interesting to note that this topic was also studied more or less at the same period by Frank Speck, with whom Stefan Samko met 25 years later in Portugal. This result and its discrete analogue immediately originated applications to various types of convolution type equations, continual and discrete, which they presented in [29], 1970; [30], [31], 1971; [37], 1973. Their papers [41], [42], 1975, adjoining this topic, concerns the study of Fredholmness of a class of convolution type equations with discontinuous symbol.

The next cycle of their research was related to singular integral equations with Carleman shift and later with a general approach to equations with the so-called involutive operator Q such that $Q^2 = I$. First, in [28], 1970, and [38], 1973, they studied the functional equation $\psi(x-\alpha) - b(x)\psi(x) = g(x)$ with the irrational (non-Carleman) shift and also with rational shift in the case of the degeneracy of the symbol of the equation in [28] and a certain boundary value problem [32], 1972, also with a non-Carleman shift.

Then they turned to SIE with Carleman type shifts, i.e.,

$$(A_1 + A_2 Q)\varphi = f \tag{4.1}$$

in the space $L^p(\Gamma)$. Here $A_i = a_i I + b_i S, i = 1, 2$, are SIOs and $Q\varphi = \varphi[\alpha(t)]$ is the Carleman shift operator, $Q^2 = I$. In their first paper [33], 1972, on such SIOs on a closed curve, they showed that the known theory of the Fredholmness of such equations is a simple consequence of the relation between the given operator $A_1 + A_2 Q$ and the so-called associate operator $A_1 - A_2 Q$. This relation is realized via the operator $U = S$, when the shift $\alpha(t)$ changes the orientation on the curve and $Uf(t) = [\alpha(t) - t]f(t)$ when the shift preserves it. They were the first to obtain the criterion of Fredholmness of SIO with Carleman shift on an open curve [36], 1972, and in the case of discontinuous coefficients $a_i(t), b_i(t)$ [39], 1973, where the results already heavily depend on the behaviour of the coefficients and the shift $\alpha(t)$ at the end-points of the curve and the points of discontinuity of the coefficients. These results were developed in [43]–[44], 1975.

In this study they realized that their approach is applicable in a more general setting of equations of form (4.1) in an abstract Banach space X with an arbitrary involutive operator Q and the linear operators A_1, A_2 in X subject to some simple axioms. Studies of such equations in an abstract form were known only in the case which reflected only the nature of singular integral equations without shifts; in an abstract form this means that the operators A_i commute with the involutive operator Q up to a compact term. They developed a general scheme to investigate the Fredholmness of such equations, not assuming the above condition. The first version of this scheme appeared in [35], 1972, where it was applied to SIE on the whole line with fractional-linear Carleman shift

$$\alpha(x) = \frac{\delta x + \beta}{\gamma x - \delta}, \tag{4.2}$$

the case of infinite curves having some peculiarities in the theory of equations with shift, the finite point $x_0 = \delta$ being shifted to the infinite point. To cover such equations, in the space of p-integrable functions, they replaced the space $L^p(\mathbb{R})$ by the weighted space $L^p(\mathbb{R}, |x - \delta|^{\frac{p}{2}-1})$ with a special weight. Later, in [52], 2000, they returned to this topic and dealt with the case of a general power weight $|x - \delta|^\gamma, -1 < \gamma < p - 1$, which however required another technique and more complicated terms in which the Fredholmness criterion is given, see also the presentation of these results in the book [53], 2001.

Then in [40], 1974, they applied this scheme to obtain the Fredholmness criterion and formula for the index for convolution equations of Wiener-Hopf type with the reflection $(Q\varphi)(t) = \varphi(\nu - t)$.

The next step of their steps was to develop such a general scheme to equations with a generalized involutive operator Q, i.e., the operator which satisfies the condition $Q^n = I$. Such a scheme was first presented in [45], 1976. In this case the initial operator

$$K = A_1 + QA_2 \cdots Q^{n-1} A_n \tag{4.3}$$

has $n-1$ associate operators $K^{(j)} = \sum\limits_{s=1}^{n} e^{2\pi i \frac{j(s-1)}{n}} Q^{s-1} A_s$ and the key moment in that paper was finding the generalization of the Gohberg-Litvinchuk matrix identity from dimension 2 to dimension n. This identity, together with some simple axioms, led them to a general statement on reducing the Fredholmness property of the operator K to that of its matrix extension without the involutive operator, which included also the formula for the index. A version of this n-dimensional matrix extension identity is also contained in [46], 1977, where it was used to obtain the criterion of Fredholmness and formula for the index in $\ell^p, 1 \leq p \leq \infty$, of discrete Wiener-Hopf equations

$$\varphi_n - \sum_{i=1}^{N} \gamma_n^i \sum_{k=0}^{\infty} a_{n-k}^i \varphi_k = f_n, \quad n = 0, 1, 2, 3, \ldots ,$$

where $\{a_n\}_{n=0}^{\infty} \in \ell^1$ and γ_n^i may stabilize at infinity to different values at infinity depending on i.

A "non-matrix" version of this general scheme for the two-terms equation $(A + QB)\varphi = f$ with an involutive operator Q of order n was developed in [47], 1977, together with applications to various concrete integral equations such as convolution type equations with reflection, singular and convolution type integral equations with complex conjugate unknown function.

This cycle of studies was presented in the book [48], 1988 (Russian version) and in its enlarged English edition [53], 2001.

5. Function spaces of fractional smoothness, influence of Steklov Mathematical Institute

As already mentioned in Section 2, in S. Samko's first studies related to one-dimensional integral equations of the first kind, Stefan Samko was interested in the description of the space of functions representable as fractional integrals of functions from a certain well-known space. Nowadays, this is well known that the range of such an operator over, for instance, a weighted Hölder space is in the scale of the same spaces, while this is not the case for Lebesgue type spaces. In the one-dimensional case he gave such a characterization of the range $I^\alpha(L^p)$ in [138], 1973, in terms of the convergence of the Marchaud fractional derivatives. An attempt to cover the multidimensional case naturally led him to the theory of fractional Sobolev type spaces and to contacts with P.I. Lizorkin, after which there started his contacts with Steklov Mathematical Institute in Moscow and his further interests and scientific activity was much influenced by the Seminar of Academician Serguey M. Nikolskii. He defended his Doctor Theses (2nd degree in the Soviet Union) in Steklov Mathematical Institute and his contacts with this institute continued in fact almost till he moved first to USA and then to Portugal.

5.1. Hypersingular integrals and spaces of the type of Riesz potentials

For the classical operator of harmonic analysis

$$I^\alpha\varphi(x) = \frac{1}{\gamma_n(\alpha)}\int\limits_{\mathbb{R}^n}\frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}, \qquad x\in\mathbb{R}^n, \tag{5.1}$$

known as the Riesz potential or Riesz fractional integral, it was well known that the inverse operator $(I^\alpha)^{-1}$ may be formally constructed via Fourier transforms. However, a direct realization of such a construction, which could be applied in this or other concrete function spaces, required a development of the apparatus of HSIs. It was undertaken by S.Samko in [144], [145], 1976, and [149], 1977. In particular, in [145] it was shown that the Marchaud-Lizorkin approach to realize the regularization of HSI via finite differences:

$$\mathbb{D}^\alpha f(x) := \int\limits_{\mathbb{R}^n}\frac{\Delta_h^\ell f(x)}{|h|^\alpha}\,dh := \lim_{\varepsilon\to 0}\mathbb{D}^\alpha_\varepsilon f(x), \quad \mathbb{D}^\alpha_\varepsilon f(x) := \int_{|h|>\varepsilon}\frac{\Delta_h^\ell f(x)}{|h|^\alpha}\,dh$$

is effective for application to potentials of Riesz type. One of the results of [145], 1976, states that such a HSI interpreted in the proper way, generates the operator left inverse to I^α in the space $L^p(\mathbb{R}^n), 1\le p<\frac{n}{\alpha}$. Moreover, the range of the operator I^α over $L^p(\mathbb{R}^n), 1<p<\frac{n}{\alpha}$ may be exactly characterized as consisting of those functions $f\in L^q(\mathbb{R}^n)$ with the Sobolev exponent $q=\frac{np}{n-\alpha p}$ for which there converge truncated HSIs $\mathbb{D}^\alpha_\varepsilon f(x)$. An extension of such a description to the case of weighted L^p-spaces with Muckenhoupt-Wheeden weights was given in [96], 1985.

In [148], 1977, a similar description of the range was shown in terms of the uniform boundedness of $\|\mathbb{D}^\alpha_{\varepsilon_k} f\|_p$ along any sequence $\varepsilon_k\to 0$. In [204], 1980, there may be found also a description of $I^\alpha(L^p)$ in similar terms via Riesz potentials of order $\{\alpha\}$ of higher derivatives of order $[\alpha]$. Results of such type were earlier known in the easier case of Bessel potential operators when one may take $q=p$. A similar characterization of the range of I^α over Orlicz spaces was given in [196], 1975.

In [146], 1977, there was suggested a certain characterization of the space of pointwise multipliers in $I^\alpha(L^p)$ in terms of the uniform L^p-boundedness of a certain family of integral operators. From this description there were derived sufficient conditions, in terms of the L^p-modulus of continuity of a function to be such a pointwise multiplier.

The study of the space of Riesz potentials led him to the introduction of new spaces

$$L^\alpha_{p,r}(\mathbb{R}^n) := \left\{f\in L^r(\mathbb{R}^n),\ \mathbb{D}^\alpha f\in L^p(\mathbb{R}^n)\right\}, \quad 1\le r<\infty,\ 1\le p<\infty,$$

of the type of fractional Sobolev spaces in which the functions f themselves and their fractional derivatives $\mathbb{D}^\alpha f$ belong to Lebesgue spaces with in general different exponents p and r, the case $r=p$ corresponding to the Bessel potential space $B^\alpha(L^p)$ and the case $r=\frac{np}{n-\alpha p}, 1<p<\frac{n}{\alpha}$, to the Riesz potential space $I^\alpha(L^p)$. It was shown in [145], 1976, and [149], 1977, that

$$L^{\alpha}_{p,r}(\mathbb{R}^n) = L^r \cap I^{\alpha}(L^p)$$

with the properly interpreted Riesz potential operator I^{α} in the overcritical case $p \geq \frac{n}{\alpha}$. One of the difficult problems to treat in such spaces was a simultaneous approximation of functions f in L^r-norms and their derivatives $\mathbb{D}^{\alpha} f$ in L^p-norms with different p and r. It was solved in [145], 1976, in the under-critical case $p < \frac{n}{\alpha}$ and in [94], 1981, in the over-critical case.

In the paper [163], 1990, where it was shown that for $f \in L^r(\mathbb{R}^n)$, the convergence in $L^p(\mathbb{R}^n)$ of the truncated HSIs $\mathbb{D}^{\alpha}_{\varepsilon} f(x)$ is equivalent to the existence of $(-\Delta)^{\alpha} f$ defined in terms of the Poisson semigroup P_{ε}:

$$(-\Delta)^{\alpha} f = \lim_{\substack{\varepsilon \to 0 \\ (L^p)}} \frac{1}{\varepsilon^{\alpha}} (I - P_{\varepsilon})^{\alpha} f \tag{5.2}$$

with p and r independent of each other, and moreover,

$$f \in L^{\alpha}_{p,r}(\mathbb{R}^n) \quad \Longleftrightarrow \quad f \in L^r(\mathbb{R}^n) \text{ and } \|(I - P_t) f\|_p \leq C t^{\alpha} \tag{5.3}$$

for all $\alpha > 0, 1 < r < \infty$ and $1 < p < \infty$.

5.2. Potential type operators with homogeneous kernels

Influenced by his interests in the theory of integral equations, he turned to applications of the method of HSI for solution of some multidimensional integral equations of the first kind. In [142], 1976, [147], 1977, and [157], 1980, he studied potential type operators

$$(K^{\alpha}_{\theta} \varphi)(x) = \int_{\mathbf{R}^n} \frac{\theta\left(\frac{x-y}{|x-y|}\right)}{|x-y|^{n-\alpha}} \varphi(y)\, dy, \quad 0 < \alpha < n, \tag{5.4}$$

with a homogeneous kernel, the function θ called the characteristics of the operator (5.4). For the symbol of this operator, i.e., the Fourier transform $\widehat{\theta_{\alpha}}(\xi)$ of the kernel $\theta_{\alpha}(x) := \frac{\theta(x')}{|x|^{n-\alpha}}$, $x' = \frac{x}{|x|}$, there was proved the formula

$$\widehat{\theta}_{\alpha}(\xi) = \Gamma(\alpha) p.f. \int\limits_{\mathbb{S}^{n-1}} \frac{\theta(\sigma)}{(-i\sigma\xi)^{\alpha}}\, d\sigma,$$

where the integral is treated in the sense of the Hadamard finite part when $\alpha \geq 1$ and it is assumed that $\theta \in C^{\lambda}$, $\lambda > \max(0, \alpha - 1)$ and $\alpha \neq 1, 2, 3, \ldots$ (this formula has a modification in the case α is an integer). In [157] there was also studied the inverse problem of constructing the function θ by the given homogeneous symbol of the potential operator.

In [157] the following theorem on the smoothness properties of the symbol was also proved:

$$\theta \in C^{\lambda}(\mathbb{S}^{n-1}) \quad \Longrightarrow \quad \widehat{\theta_{\alpha}} \in C^{\lambda-\alpha+1}(\mathbb{S}^{n-1})$$

where the Hölder space $C^{\lambda-\alpha+1}(\mathbb{S}^{n-1})$ should be introduced with the logarithmic factor in the case of integer α. (Note that this corresponds to the limiting case $p=\infty$, if compared with a setting of smoothness in $L_p^\lambda(\mathbb{S}^{n-1})$-terms.)

The following result was also proved:

$$\theta \in C^\lambda(\mathbb{S}^{n-1}),\ \lambda \geq \alpha+n-1 \quad \Longrightarrow \quad K_\theta^\alpha(L^p) \subset I^\alpha(L^p),\ 1<p<\frac{n}{\alpha}$$

and $K_\theta^\alpha(L^p) = I^\alpha(L^p)$ in the elliptic case.

To invert potential type operators of form K_θ^α, he introduced the generalized HSI of the form

$$\mathbb{D}_\Omega^\alpha f)(x) = \int\limits_{\mathbf{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} \Omega\left(\frac{y}{|y|}\right) dy$$

and showed that in application to such an inversion the choice of the type of the finite difference $(\Delta_y^\ell f)(x)$ plays an important role, especially in the case where α is an integer. It was proved in the elliptic case, that for sufficiently smooth $\theta(x')$, under the appropriate choice of the type of the difference, there exists an Ω, such that $\mathbb{D}_\Omega^\alpha$ is the left inverse to K_θ^α in the space $L^p(\mathbb{R}^n)$ and the function Ω may be explicitly constructed:

$$\Omega(x') = \text{const}\, p.f. \int\limits_{\mathbb{S}^{n-1}} \frac{d\sigma}{\widehat{\theta_\alpha}(ix'\sigma)^{n+\alpha}}.$$

In a special case where $\theta(x') = P_2(x',x')$ is a restriction of a quadratic form onto $\mathbb{S}^{n-1}$, in [167], 1993, the above integral was explicitly calculated supposing that there are known the eigenvalues of the matrix P and the matrix W transforming it to the diagonal form.

As is well known, various potential kernels serve as fundamental solutions to differential operators in partial derivatives. As a general statement of such a kind, in [153], 1978, there was proved that *every homogeneous differential operator with constant coefficients may be represented in the form of a HSI with a certain function Ω, which is also a homogeneous polynomial of the same order and it is constructed explicitly.*

As a by-product of his studies of potential operators with homogeneous kernel, in [156], 1978, he gave a formula for the Fourier transform of functions of the form $Y_m(x)|x|^{-\beta}$, where Y_m is a harmonic polynomial, for some "exceptional" values of γ, and in [152], 1978, he gave the solution of the following problem: given a homogeneous polynomial

$$P_m(x) = \sum_{|j|=m} a_j x^j, \quad x^j = x_1^{j_1} \cdots x_n^{j_n},$$

what are necessary and sufficient conditions (imposed on the coefficients a_j) for $P_m(x)$ to be harmonic?

The problem of convergence of HSI with a homogeneous function $\Omega(x')$ was studied in [95], 1982, where it was shown that if for a function f there converges its HSI with $\Omega \equiv \text{const}$, then it converges with any bounded Ω; the inverse is also true under a certain ellipticity condition on Ω. Similar questions in the case of non-homogeneous functions Ω stabilizing at the origin and infinity, were earlier studied in [149], 1977.

An application of the method of HSI to the inversion of potential type operators with a non-homogeneous radial characteristic $\theta(|x-y|)$ may be found in [206], 1985.

We also refer to an overview of applications of this method to potential-type integral transforms in the paper [166], 1993.

In contrast to the case of Bessel type potentials, Riesz fractional integrals $I^\alpha f$ when interpreted within the frameworks of distributions, need a test functions space invariant with respect to the operator I^α. Such a space of Schwartz test functions which are orthogonal to all the polynomials, is known as the Lizorkin test function space. In relation with studies of potential type operators whose symbol may be zero or infinity on a set different from the origin, S. Samko in [151], 1977, introduced a Lizorkin type space of Schwartz test functions whose Fourier transforms vanish on a given set $V \subset \mathbb{R}^n$:

$$\Phi_V := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \ \widehat{\varphi}(\xi) = 0 \text{ for } \xi \in V\} \tag{5.5}$$

and gave their description in the case of conic sets V corresponding to convolution operators with homogeneous kernel. For a large variety of sets V with $|V| = 0$, in [158], 1982, he proved that such a test function space is dense in $L^p(\mathbb{R}^n), 1 < p < \infty$, which proved to be essential in many papers on inverting potential type operators, where it was necessary to approximate by nice functions from a space invariant with respect to the operator. See also [168], 1995, for similar denseness in the spaces $L^{\bar{p}}(\mathbb{R}^n)$ with mixed norm.

5.3. Spherical HSI and potentials

After the investigation of symbols of convolution operators with homogeneous kernel, and also influenced by the studies of P.I. Lizorkin and S.M. Nikolskii on approximation of functions on the sphere $\mathbb{S}^{n-1}$, S. Samko investigated potential operators on $\mathbb{S}^{n-1}$; in [150], 1977, he introduced some new forms of spherical fractional type integrals and HSI. One of them proved to have an important role in his paper [102], 2010, more than 30 years later, where such a construction of a spherical potential operator arose in application to an inverse problem of aerodynamics. In the paper [159], 1983, he presented a survey on spherical singular and potential operators, including the related problem of restoring the kernel of the spatial potential by its symbol. In [100] there was given a characterization of the fractional Sobolev type space $L^\alpha_p(\mathbb{S}^{n-1})$ of functions $f \in L_p(\mathbb{S}^{n-1})$ such that the fractional power of the Beltrami-Laplace operator is also in $L_p(\mathbb{S}^{n-1})$, via convergence of the spherical HSIs

$$\int\limits_{\mathbb{S}^{n-1}} \frac{f(x)-f(\sigma)}{|x-\sigma|^{n-1-\alpha}} d\sigma, \qquad x \in \mathbb{S}^{n-1}, \ \ 0<\alpha<1.$$