Computational Noncommutative Algebra and Applications

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Dedication

This book is dedicated to the cherished memory of Edward Joseph Sanchas, a most decent and trusted human being whom we always counted on to take charge of any and all problems. He was a great man. We miss him deeply.

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Preface

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Jim Byrnes

The chapters in this volume were presented at the July 2003 NATO Advanced Study Institute on *Computational Noncommutative Algebra and Applications*. The conference was held at the beautiful Il Ciocco resort near Lucca, in the glorious Tuscany region of northern Italy. Once again we gathered at this idyllic spot to explore and extend the reciprocity between mathematics and engineering. The dynamic interaction between world-renowned scientists from the usually disparate communities of pure mathematicians and applied scientists, which occurred at our 1989, 1991, 1992, 1998, and 2000 ASI's, continued at this meeting.

The fusion of algebra, analysis and geometry, and their application to real world problems, have been dominant themes underlying mathematics for over a century. Geometric algebras, introduced and classified by Clifford in the late 19th century, have played a prominent role in this effort, as seen in the mathematical work of Cartan, Brauer, Weyl, Chevelley, Atiyah, and Bott, and in applications to physics in the work of Pauli, Dirac and others. One of the most important applications of geometric algebras to geometry is to the representation of groups of Euclidean and Minkowski rotations. This aspect and its direct relation to robotics and vision were discussed by several of the Principal Lecturers, and are covered in this book.

Moreover, group theory, beginning with the work of Burnside, Frobenius and Schur, has been influenced by even more general problems. As a result, general group actions have provided the setting for powerful methods within group theory and for the use of groups in applications to physics, chemistry, molecular biology, and signal processing. These aspects, too, are covered in what follows.

With the rapidly growing importance of, and ever expanding conceptual and computational demands on signal and image processing in remote sensing, computer vision, medical image processing, and biological signal processing, and on neural and quantum computing, geometric algebras, and computational group harmonic analysis, the topics of the following chapters have emerged as key tools. The authors include many of the world's leading experts in the development of new algebraic modeling and signal representation methodologies, novel Fourier-based and geometric transforms, and computational algorithms required for realizing the potential of these new application fields.

The ASI brought together these world leaders from both academia and industry, with extensive multidisciplinary backgrounds evidenced by their research and participation in numerous workshops and conferences. This created an interactive forum for initiating new and intensifying existing efforts aimed at creating a unified computational noncommutative algebra for advancing the broad range of applications indicated above. The forum provided opportunities for young scientists and engineers to learn more about these problem areas, and the vital role played by new mathematical insights, from the recognized experts in this vital and growing area of both pure and applied science.

The talks and the following chapters were designed to address an audience consisting of a broad spectrum of scientists, engineers, and mathematicians involved in these fields. Participants had the opportunity to interact with those individuals who have been on the forefront of the ongoing explosion of work in computational noncommutative algebra, to learn firsthand the details and subtleties of this exciting area, and to hear these experts discuss in accessible terms their contributions and ideas for future research. This volume offers these insights to those who were unable to attend.

The cooperation of many individuals and organizations was required in order to make the conference the success that it was. First and foremost I wish to thank NATO, and especially Dr. F. Pedrazzini and his most able assistant, Ms. Alison Trapp, for the initial grant and subsequent help. Financial support was also received from the Air Force Office of Scientific Research (Dr. Jon Sjogren), the Defense Advanced Research Projects Agency (Dr. Douglas Cochran), the National Science Foundation (Dr. Sylvia Wiegand), USAF Rome Laboratories (Drs. Michael Wicks, Braham Himed, and Gerard Genello), US Army SMDC (Drs. Pete Kirkland and Robert McMillan), EOARD (Dr. Chris Reuter), the Raytheon Company (Dr. David N. Martin), Bielefeld University (Professor Andreas Dress), Bonn University (Professor Michael Clausen), Melbourne University, and Prometheus Inc. This additional support is gratefully acknowledged.

I wish to express my sincere appreciation to my assistants, Marcia Byrnes and Gerald Ostheimer, for their invaluable aid. I am also grateful to Kathryn Hargreaves, our TEXnician, for her superlative work in preparing this volume. Finally, my heartfelt thanks to the Il Ciocco staff, especially Bruno Giannasi and Alberto Suffredini, for offering an ideal setting, not to mention the magnificent meals, that promoted the productive interaction between the participants of the conference. All of the above, the speakers, and the remaining conferees, made it possible for our Advanced Study Institute, and this volume, to fulfill the stated NATO objectives of disseminating advanced knowledge and fostering international scientific contacts.

September 22, 2003 *Jim Byrnes*, Newport, Rhode Island

Acknowledgments

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CLIFFORD GEOMETRIC ALGEBRAS IN MULTILINEAR ALGEBRA AND NON-EUCLIDEAN GEOMETRIES

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Abstract Given a quadratic form on a vector space, the geometric algebra of the corresponding pseudo-euclidean space is defined in terms of a simple set of rules which characterizes the geometric product of vectors. We develop geometric algebra in such a way that it augments, but remains fully compatible with, the more traditional tools of matrix algebra. Indeed, matrix multiplication arises naturally from the geometric multiplication of vectors by introducing a spectral basis of mutually annihiliating idempotents in the geometric algebra. With the help of a few more algebraic identities, and given the proper geometric interpretation, the geometric algebra can be applied to the study of affine, projective, conformal and other geometries. The advantage of geometric algebra is that it provides a single algebraic framework with a comprehensive, but flexible, geometric interpretation. For example, the affine plane of rays is obtained from the euclidean plane of points by adding a single anti-commuting vector to the underlying vector space. The key to the study of noneuclidean geometries is the definition of the operations of meet and join, in terms of which incidence relationships are expressed. The horosphere provides a homogeneous model of euclidean space, and is obtained by adding a second anti-commuting vector to the underlying vector space of the affine plane. Linear orthogonal transformations on the higher dimensional vector space correspond to conformal or Möbius transformations on the horosphere. The horosphere was first constructed by F.A. Wachter (1792–1817), but has only recently attracted attention by offering a host of new computational tools

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in projective and hyperbolic geometries when formulated in terms of geometric algebra.

Keywords: affine geometry, Clifford algebra, conformal geometry, conformal group, euclidean geometry, geometric algebra, horosphere, Möbius transformation, non-euclidean geometry, projective geometry, spectral decomposition.

1. Geometric algebra

A Geometric algebra is generated by taking linear combinations of geometric products of vectors in a vector space taken together with a specified bilinear form. Here we shall study the geometric algebras of the *pseudo-euclidean* vector spaces $\mathcal{G}_{p,q} := \mathcal{G}_{p,q}(I\!R^{p,q})$ for which we have the indefinite metric

$$x \cdot y = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j$$

for $x = \begin{pmatrix} x_1 & \cdots & x_{p+q} \end{pmatrix}$ and $y = \begin{pmatrix} y_1 & \cdots & y_{p+q} \end{pmatrix}$ in $I\!R^{p,q}$. We first study the geometric algebra of the more familiar Euclidean space.

1.1 Geometric algebra of Euclidean n-space

We begin by introducing the geometric algebra $\mathcal{G}_n := \mathcal{G}(I\!R^n)$ of the familiar Euclidean n-space

$$I\!R^n = \{x |\ x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \text{ for } x_i \in I\!R\}.$$

Recall the dual interpretations of each element $x \in I\!R^n$, both as a point of $I\!R^n$ with the coordinates $\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and as the position vector or *directed line segment* from the origin to the point. We can thus express each vector $x \in I\!R^n$ as a linear combination of the *standard orthonormal basis vectors* $\{e_1, e_2, \cdots, e_n\}$ where $e_i = \begin{pmatrix} 0 & \cdots & 0 & 1_i & 0 & \cdots & 0 \end{pmatrix}$, namely

$$x = \sum_{i=1}^{n} x_i e_i.$$

The vectors of $I\!R^n$ are added and multiplied by scalars in the usual way, and the positive definite *inner product* of the vectors x and $y = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$ is given by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i. \tag{1}$$

The geometric algebra $\mathcal{G}_n$ is generated by the *geometric multiplication* and addition of vectors in $I\!R^n$. In order to efficiently introduce the geometric product of vectors, we note that the resulting geometric algebra $\mathcal{G}_n$ is isomorphic to an appropriate matrix algebra under addition and geometric multiplication. Thus, like matrix algebra, $\mathcal{G}_n$ is an associative, but non-commutative algebra, but unlike matrix algebra the elements of $\mathcal{G}_n$ are assigned a comprehensive geometric interpretation. The two fundamental rules governing geometric multiplication and its interpretation are:

- For each vector $x \in I\!R^n$,

$$x^2 = xx = |x|^2 = \sum_{i=1}^{n} x_i^2 \tag{2}$$

  where $|x|$ is the usual *Euclidean norm* of the vector x.

- If $a_1, a_2, \dots, a_k \in I\!R^n$ are k mutually orthogonal vectors, then the product

$$A_k = a_1 a_2 \dots a_k \tag{3}$$

  is totally antisymmetric and has the geometric interpretation of a *simple k-vector* or a *directed k-plane* .[1]

Let us explore some of the many consequences of these two basic rules. Applying the first rule (2) to the sum $a + b$ of the vectors $a, b \in I\!R^2$, we get

$$(a+b)^2 = a^2 + ab + ba + b^2,$$

or

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}(|a+b|^2 - |a|^2 - |b|^2)$$

which is a statement of the famous *law of cosines* . In the special case when the vectors a and b are orthogonal, and therefore anticommutative by the second rule (3), we have $ab = -ba$ and $a \cdot b = 0$.

If we multiply the orthonormal basis vectors $e_{12} := e_1 e_2$, we get the 2-vector or *bivector* e_{12}, pictured as the *directed plane segment* in Figure 1. Note that the *orientation* of the bivector e_{12} is counterclockwise, and that the bivector $e_{21} := e_2 e_1 = -e_1 e_2 = -e_{12}$ has the opposite or clockwise orientation.

[1]This means that the product changes its sign under the interchange of any two of the orthogonal vectors in its argument.

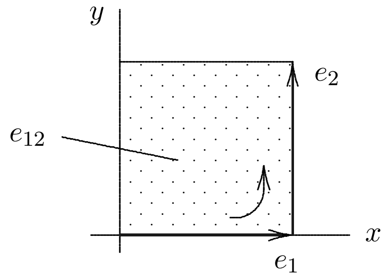

Figure 1. The directed plane segment $e_{12} = e_1 e_2$.

We can now write down an orthonormal basis for the geometric algebra $\mathcal{G}_n$, generated by the orthonormal basis vectors $\{e_i |\ 1 \le i \le n\}$. In terms of the modified cartesian-like product, $\times_{i=1}^{n}(1, e_i) :=$

$$\{1, e_1, \dots, e_n, e_{12}, \dots, e_{(n-1)n}, \dots, \dots, e_{1\cdots(n-1)}, \dots, e_{2\cdots n}, e_{1\dots n}\}.$$

There are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

linearly independent elements in the standard orthonormal basis of $\mathcal{G}_n$. Any *multivector* or *geometric number* $g \in \mathcal{G}_n$ can be expressed as a sum of its homogeneous k-vector parts,

$$g = g_0 + \cdots + g_k + \cdots + g_n$$

where $g_k :=< g >_k = \sum_\sigma \alpha_\sigma e_\sigma$ where $\sigma = \sigma_1 \cdots \sigma_k$ for $1 \le \sigma_1 < \cdots < \sigma_k \le n$, and $\alpha_\sigma \in I\!R$. The real part $g_0 :=< g >_0 = \alpha_0 e_0 = \alpha_0$ of the geometric number g is just a real number, since $e_0 := 1$. By *definition*, any k-vector can be written as a linear combination of simple k-vectors or k-*blades* , [8, p.4].

Given two vectors $a, b \in I\!R^n$, we can decompose the vector a into components parallel and perpendicular to b, $a = a_\| + a_\perp$, where

$$a_\| = (a \cdot b)\frac{b}{|b|^2} = (a \cdot b)b^{-1},$$

and $a_\perp := a - a_\|$, see Figure 2.

With the help of (3), we now calculate the geometric product of the vectors a and b, getting

$$ab = (a_\| + a_\perp)b = a_\| \cdot b + a_\perp \wedge b = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \qquad (4)$$

$$a_{\|} = (a \cdot b)\frac{b}{|b|^2} = (a \cdot b)b^{-1}.$$

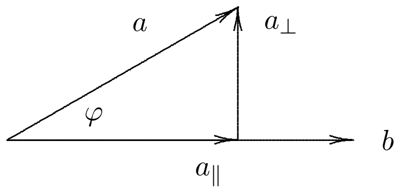

Figure 2. Decomposition of a into parallel and perpendicular parts.

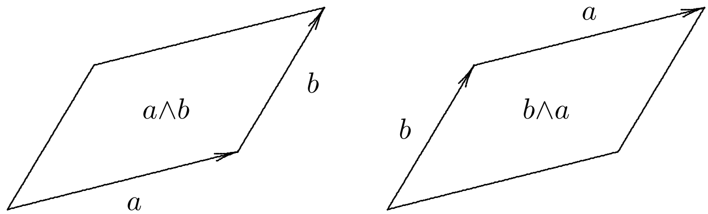

Figure 3. The bivectors $a \wedge b$ and $b \wedge a$.

where the *outer product* $a \wedge b := \frac{1}{2}(ab - ba) = a_{\perp}b = -ba_{\perp} = -b \wedge a$ is the bivector shown in Figure 3. The basic formula (4) shows that the geometric product ab is the sum of a scalar and a bivector part which characterizes the relative directions of a and b. If we make the assumption that a and b lie in the plane of the bivector e_{12}, then we can write

$$ab = |a||b|(\cos\varphi + I \sin\varphi) = |a||b|e^{I\varphi}, \tag{5}$$

where $I := e_{12} = e_1 e_2$ has the familiar property that

$$I^2 = e_1 e_2 e_1 e_2 = -e_1 e_2 e_2 e_1 = -e_1^2 e_2^2 = -1.$$

Equation (5) is the *Euler formula* for the geometric multiplication of vectors.

The definition of the inner product $a \cdot b$ and outer product $a \wedge b$ can be easily extended to $a \cdot B_r$ and $a \wedge B_r$, respectively, where $r \geq 0$ denotes the *grade* of the r-vector B_r:

DEFINITION 2 *The inner product or contraction $a \cdot B_r$ of a vector a with an r-vector B_r is determined by*

$$a \cdot B_r = \frac{1}{2}(aB_r + (-1)^{r+1}B_r a) = (-1)^{r+1}B_r \cdot a.$$

DEFINITION 3 *The outer product $a \wedge B_r$ of a vector a with an r-vector B_r is determined by*

$$a \wedge B_r = \frac{1}{2}(aB_r - (-1)^{r+1}B_r a) = -(-1)^{r+1}B_r \wedge a.$$

Note that $a \cdot \beta = \beta \cdot a = 0$ and $a \wedge \beta = \beta \wedge a = \beta a$ for the scalar $\beta \in \mathbb{R}$. Indeed, we will soon show that $a \cdot B_r =< aB_r >_{r-1}$ and $a \wedge B_r =< aB_r >_{r+1}$ for all $r \geq 1$; we have already seen that this is true when $r = 1$. There are different conventions regarding the use of the dot product and contraction [5, p. 35].

One of the most basic geometric algebras is the geometric algebra $\mathcal{G}_3$ of 3 dimensional Euclidean space which we live in. The complete standard orthonormal basis of this geometric algebra is

$$\mathcal{G}_3 = \times_{i=1}^{3}(1, e_i) = span\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}.$$

Any geometric number $g \in \mathcal{G}_3$ has the form $g = \alpha + v_1 + iv_2 + \beta i$ where $i := e_{123}$. Notice that we have expressed the bivector part of g as the *dual* of the vector v_2. Thus the geometric number $g = (\alpha + i\beta) + (v_1 + iv_2)$ can be expressed as the sum of its *complex scalar part* $(\alpha + i\beta)$ and a *complex vector part* $(v_1 + iv_2)$. Note that the complex scalar part has all the properties of an ordinary complex number $z = x + iy$. This follows easily from the fact that the pseudoscalar $i = e_{123}$ satisfies $i^2 = e_{123}e_{123} = e_{23}e_{23} = -1$.

We can use the Euler form (5) to see that

$$a = a(b^{-1}b) = (ab^{-1})b = \left(\frac{ab}{b^2}\right)b,$$

so the geometric quantity $ab/|b|^2$ *rotates and dilates* the vector b into the vector *a when multiplied by b on the left.* Similarly, multiplying b on right by $\frac{ba}{b^2}$ also rotates and dilates the vector b into the vector a. By re-expressing this result in terms of the Euler angle φ, letting $I = ie_3$, and assuming that $|a| = |b|$, we can write $a = \exp(ie_3\varphi)b = b\exp(-ie_3\varphi)$. Even more powerfully, and more generally, we can write

$$a = \exp(I\varphi/2)b\exp(-I\varphi/2),$$

which expresses the $\frac{1}{2}$-angle formula for rotating the vector $b \in \mathbb{R}^n$ in the plane of the simple bivector I through the angle φ. There are many more formulas for expressing reflexions and rotations in $\mathbb{R}^n$, or in the pseudo-euclidean spaces $\mathbb{R}^{p,q}$, [8], [10].

3.1 Basic algebraic identities

One of the most difficult aspects of learning geometric algebra is coming to terms with a host of unfamiliar algebraic identities. These important identities can be quickly mastered if they are established in a careful systematic way. The most important of these identities follows