ALGEBRA

A TEXT-BOOK OF DETERMINANTS, MATRICES, AND ALGEBRAIC FORMS

BY

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PREFACE TO SECOND EDITION

In revising the book for a second edition I have corrected a number of slips and misprints and I have added a new chapter on latent vectors.

W. L. F.

HERTFORD COLLEGE, OXFORD,
1957.

PREFACE TO FIRST EDITION

IN writing this book I have tried to provide a text-book of the more elementary properties of determinants, matrices, and algebraic forms. The sections on determinants and matrices, Parts I and II of the book, are, to some extent, suitable either for undergraduates or for boys in their last year at school. Part III is suitable for study at a university and is not intended to be read at school.

The book as a whole is written primarily for undergraduates. University teaching in mathematics should, in my view, provide at least two things. The first is a broad basis of knowledge comprising such theories and theorems in any one branch of mathematics as are of constant application in other branches. The second is incentive and opportunity to acquire a detailed knowledge of some one branch of mathematics. The books available make reasonable provision for the latter, especially if the student has, as he should have, a working knowledge of at least one foreign language. But we are deplorably lacking in books that cut down each topic, I will not say to a minimum, but to something that may reasonably be reckoned as an essential part of an undergraduate's mathematical education.

Accordingly, I have written this book on the same general plan as that adopted in my book on convergence. I have included topics commonly required for a university honours course in pure and applied mathematics: I have excluded topics appropriate to post-graduate or to highly specialized courses of study.

Some of the books to which I am indebted may well serve as a guide for my readers to further algebraic reading. Without pretending that the list exhausts my indebtedness to others, I may note the following: Scott and Matthews, *Theory of Determinants*; Bôcher, *Introduction to Higher Algebra*; Dickson, *Modern Algebraic Theories*; Aitken, *Determinants and Matrices*; Turnbull, *The Theory of Determinants, Matrices, and Invariants*; Elliott, *Introduction to the Algebra of Quantics*; Turnbull and Aitken, *The Theory of Canonical Matrices*; Salmon, *Modern*

Higher Algebra (though the 'modern' refers to some sixty years back); Burnside and Panton, *Theory of Equations*. Further, though the reference will be useless to my readers, I gratefully acknowledge my debt to Professor E. T. Whittaker, whose invaluable 'research lectures' on matrices I studied at Edinburgh many years ago.

The omissions from this book are many. I hope they are, all of them, deliberate. It would have been easy to fit in something about the theory of equations and eliminants, or to digress at one of several possible points in order to introduce the notion of a group, or to enlarge upon number rings and fields so as to give some hint of modern abstract algebra. A book written expressly for undergraduates and dealing with one or more of these topics would be a valuable addition to our stock of university text-books, but I think little is to be gained by references to such subjects when it is not intended to develop them seriously.

In part, the book was read, while still in manuscript, by my friend and colleague, the late Mr. J. Hodgkinson, whose excellent lectures on Algebra will be remembered by many Oxford men. In the exacting task of reading proofs and checking references I have again received invaluable help from Professor E. T. Copson, who has read all the proofs once and checked nearly all the examples. I am deeply grateful to him for this work and, in particular, for the criticisms which have enabled me to remove some notable faults from the text.

Finally, I wish to thank the staff of the University Press, both on its publishing and its printing side, for their excellent work on this book. I have been concerned with the printing of mathematical work (mostly that of other people!) for many years, and I still marvel at the patience and skill that go to the printing of a mathematical book or periodical.

W. L. F.

HERTFORD COLLEGE, OXFORD,
September 1940.

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PART I

PRELIMINARY NOTE; CHAPTERS ON DETERMINANTS

PRELIMINARY NOTE

1. Number

In its initial stages algebra is little more than a generalization of elementary arithmetic. It deals only with the positive integers, 1, 2, 3,.... We can all remember the type of problem that began 'let x be the number of eggs', and if x came to $3\frac{1}{2}$ we knew we were wrong.

In later stages x is permitted to be negative or zero, to be the ratio of two integers, and then to be any real number either rational, such as $3\frac{1}{3}$ or $-\frac{1}{4}$, or irrational, such as π or $\sqrt{3}$. Finally, with the solution of the quadratic equation, x is permitted to be a complex number, such as $2+3i$.

The numbers used in this book may be either real or complex and we shall assume that readers have studied, to a greater or a lesser extent, the precise definitions of these numbers and the rules governing their addition, subtraction, multiplication, and division.

2. Number rings

Consider the set of numbers

$$0, \ \pm 1, \ \pm 2, \ \dots . \tag{1}$$

Let r, s denote numbers selected from (1). Then, whether r and s denote the same or different numbers, the numbers

$$r+s, \quad r-s, \quad r\times s$$

all belong to (1). This property of the set (1) is shared by other sets of numbers. For example,

$$\text{all numbers of the form } a+b\sqrt{5}, \tag{2}$$

where a and b belong to (1), have the same property; if r and s belong to (2), then so do $r+s$, $r-s$, and $r\times s$. A set of numbers having this property is called a RING of numbers.

3. Number fields

3.1. Consider the set of numbers comprising 0 and every number of the form p/q, where p and q belong to (1) and q is not zero, that is to say,

$$\text{the set of all rational real numbers.} \qquad (3)$$

Let r, s denote numbers selected from (3). Then, when s is not zero, whether r and s denote the same or different numbers, the numbers

$$r+s, \quad r-s, \quad r\times s, \quad r\div s$$

all belong to the set (3).

This property characterizes what is called a FIELD of numbers. The property is shared by the following sets, among many others:—

the set of all complex numbers; (4)

the set of all real numbers (rational and irrational); (5)

the set of all numbers of the form $p+q\sqrt{3}$, where p and q belong to (3). (6)

Each of the sets (4), (5), and (6) constitutes a field.

DEFINITION. *A set of numbers, real or complex, is said to form a* FIELD OF NUMBERS *when, if r and s belong to the set and s is not zero,*

$$r+s, \quad r-s, \quad r\times s, \quad r\div s$$

also belong to the set.

Notice that the set (1) is not a field; for, whereas it contains the numbers 1 and 2, it does not contain the number $\frac{1}{2}$.

3.2. Most of the propositions in this book presuppose that the work is carried out within *a* field of numbers; what *particular* field is usually of little consequence.

In the early part of the book this aspect of the matter need not be emphasized: in some of the later chapters the essence of the theorem is that all the operations envisaged by the theorem can be carried out within the confines of any given field of numbers.

In this preliminary note we wish to do no more than give a formal definition of a field of numbers and to familiarize the reader with the concept.

4. Matrices

A set of mn numbers, real or complex, arranged in an array of m columns and n rows is called a matrix. Thus

$$\begin{matrix} a_{11} & a_{12} & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & a_{2m} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nm} \end{matrix}$$

is a matrix. When $m = n$ we speak of a square matrix of order n.

Associated with any given square matrix of order n there are a number of algebraical entities. The matrix written above, with $m = n$, is associated

(i) with the determinant

$$\begin{vmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{vmatrix};$$

(ii) with the form

$$\sum_{r=1}^{n} \sum_{s=1}^{n} a_{rs} x_r x_s,$$

of degree 2 in the n variables $x_1, x_2, \ldots, x_n$;

(iii) with the bilinear form

$$\sum_{r=1}^{n} \sum_{s=1}^{n} a_{rs} x_r y_s$$

in the $2n$ variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$;

(iv) with the Hermitian form

$$\sum_{r=1}^{n} \sum_{s=1}^{n} a_{rs} x_r \bar{x}_s,$$

where x_s and $\bar{x}_s$ are conjugate complex numbers;

(v) with the linear transformations

$$x_r = \sum_{s=1}^{n} a_{rs} X_s \quad (r = 1, \ldots, n),$$

$$L_s = \sum_{r=1}^{n} a_{rs} l_r \quad (s = 1, \ldots, n).$$

The theories of the matrix and of its associated forms are closely knit together. The plan of expounding these theories that I have adopted is, roughly, this: Part I develops properties of the determinant; Part II develops the algebra of matrices, referring back to Part I for any result about determinants that may be needed; Part III develops the theory of the other associated forms.

CHAPTER I

ELEMENTARY PROPERTIES OF DETERMINANTS

1. Introduction

1.1. In the following chapters it is assumed that most readers will already be familiar with determinants of the second and third orders. On the other hand, no theorems about such determinants are assumed, so that the account given here is complete in itself.

Until the middle of the last century the use of determinant notation was practically unknown, but once introduced it gained such popularity that it is now employed in almost every branch of mathematics. The theory has been developed to such an extent that few mathematicians would pretend to a knowledge of the whole of it. On the other hand, the range of theory that is of constant application in other branches of mathematics is relatively small, and it is this restricted range that the book covers.

1.2. Determinants of the second and third orders. Determinants are, in origin, closely connected with the solution of linear equations.

Suppose that the two equations

$$a_1 x+b_1 y = 0, \qquad a_2 x+b_2 y = 0$$

are satisfied by a pair of numbers x and y, one of them, at least, being different from zero. Then

$$b_2(a_1 x+b_1 y)-b_1(a_2 x+b_2 y) = 0,$$

and so $$(a_1 b_2-a_2 b_1)x = 0.$$

Similarly, $(a_1 b_2-a_2 b_1)y = 0$, and so $a_1 b_2-a_2 b_1 = 0$.

The number $a_1 b_2-a_2 b_1$ is a simple example of a determinant; it is usually written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{1}$$

The term $a_1 b_2$ is referred to as 'the leading diagonal'. Since there are two rows and two columns, the determinant is said to be 'of order two', or 'of the second order'.

The determinant has one obvious property. If, in (1), we interchange simultaneously a_1 and b_1, a_2 and b_2, we get

$$b_1 a_2 - b_2 a_1 \quad \text{instead of} \quad a_1 b_2 - a_2 b_1.$$

That is, the interchange of two columns of (1) reproduces the same terms, namely $a_1 b_2$ and $a_2 b_1$, but in a different order and with the opposite signs.

Again, let numbers x, y, and z, not all zero, satisfy the three equations

$$a_1 x + b_1 y + c_1 z = 0, \tag{2}$$

$$a_2 x + b_2 y + c_2 z = 0, \tag{3}$$

$$a_3 x + b_3 y + c_3 z = 0; \tag{4}$$

then, from equations (3) and (4),

$$(a_2 b_3 - a_3 b_2)x - (b_2 c_3 - b_3 c_2)z = 0,$$

$$(a_2 b_3 - a_3 b_2)y - (c_2 a_3 - c_3 a_2)z = 0,$$

and so, from equation (2),

$$z\{a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2)\} = 0.$$

We denote the coefficient of z, which may be written as

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1,$$

by Δ; so that our result is $z\Delta = 0$.

By similar working we can show that

$$x\Delta = 0, \qquad y\Delta = 0.$$

Since x, y, and z are not all zero, $\Delta = 0$.

The number Δ is usually written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

in which form it is referred to as a 'determinant of order three' or a 'determinant of the third order'. The term $a_1 b_2 c_3$ is referred to as 'the leading diagonal'.

1.3. It is thus suggested that, associated with n linear equations in n variables, say

$$a_1 x + b_1 y + \ldots + k_1 z = 0,$$

$$a_2 x + b_2 y + \ldots + k_2 z = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n x + b_n y + \ldots + k_n z = 0,$$

there is a certain function of the coefficients which must be zero if all the equations are to be satisfied by a set of values x, y,..., z which are not all zero. It is suggested that this function of the coefficients may be conveniently denoted by

$$\begin{vmatrix} a_1 & b_1 & . & . & k_1 \\ a_2 & b_2 & . & . & k_2 \\ . & . & . & . & . \\ a_n & b_n & . & . & k_n \end{vmatrix},$$

in which form it may be referred to as a determinant of order n, and $a_1\, b_2 \ldots k_n$ called the leading diagonal.

Just as we formed a determinant of order three (in § 1.2) by using determinants of order two, so we could form a determinant of order four by using those of order three, and proceed step by step to a definition of a determinant of order n. But this is not the only possible procedure and we shall arrive at our definition by another path.

We shall first observe certain properties of determinants of the third order and then define a determinant of order n in such a way that these properties are preserved for determinants of every order.

1.4. Note on definitions. There are many different ways of defining a determinant of order n, though all the definitions lead to the same result in the end. The only particular merit we claim for our own definition is that it is easily reconcilable with any of the others, and so makes reference to other books a simple matter.

1.5. Properties of determinants of order three. As we have seen in § 1.2, the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

stands for the expression

$$+a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1. \quad (1)$$

The following facts are all but self-evident:

(I) The expression (1) is of the form

$$\sum \pm a_r b_s c_t,$$

wherein the sum is taken over the six possible ways of assigning to r, s, t the values 1, 2, 3 in some order and without repetition.

(II) The leading diagonal term $a_1 b_2 c_3$ is prefixed by $+$.

(III) As with the determinant of order 2 (§ 1.2), the interchange of any two letters throughout the expression (1) reproduces the same set of terms, but in a different order and with the opposite signs prefixed to them. For example, when a and b are interchanged† in (1), we get

$$+b_1 a_2 c_3 - b_1 a_3 c_2 + b_2 a_3 c_1 - b_2 a_1 c_3 + b_3 a_1 c_2 - b_3 a_2 c_1,$$

which consists of the terms of (1), but in a different order and with the opposite signs prefixed.

2. Determinants of order n

2.1. Having observed (§ 1.5) three essential properties of a determinant of the third order, we now define a determinant of order n.

DEFINITION. *The determinant*

$$\Delta_n \equiv \begin{vmatrix} a_1 & b_1 & . & . & j_1 & k_1 \\ a_2 & b_2 & . & . & j_2 & k_2 \\ . & . & . & . & . & . \\ a_n & b_n & . & . & j_n & k_n \end{vmatrix} \quad (1)$$

is that function of the a's, b's,..., k's which satisfies the three conditions:

(I) *it is an expression of the form*

$$\sum \pm a_r b_s \dots k_\theta, \quad (2)$$

wherein the sum is taken over the $n!$ possible ways of assigning to r, s,..., θ the values 1, 2,..., n in some order, and without repetition;

(II) *the leading diagonal term, $a_1 b_2 \dots k_n$, is prefixed by the sign $+$;*

(III) *the sign prefixed to any other term is such that the interchange of any two letters‡ throughout* (2) *reproduces the same*

† Throughout we use the phrase 'interchange a and b' to denote the simultaneous interchanges a_1 and b_1, a_2 and b_2, a_3 and b_3, ..., a_n and b_n.

‡ See previous footnote. The interchange of p and q, say, means the simultaneous interchanges

$$p_1 \text{ and } q_1, p_2 \text{ and } q_2, \dots, p_n \text{ and } q_n.$$

set of terms, but in a different order of occurrence, and with the opposite signs prefixed.

Before proceeding we must prove that the definition yields one function of the a's, b's,..., k's and one only. The proof that follows is divided into four main steps.

First step. Let the letters $a, b, \ldots, k$, which correspond to the columns of (1), be written down in any order, say

$$\ldots,\ d,\ g,\ a,\ \ldots,\ p,\ \ldots,\ q,\ \ldots. \tag{A}$$

An interchange of two letters that stand next to each other is called an ADJACENT INTERCHANGE. Take any two letters p and q, having, say, m letters between them in the order (A). By $m+1$ adjacent interchanges, in each of which p is moved one place to the right, we reach a stage at which p comes next after q; by m further adjacent interchanges, in each of which q is moved one place to the left, we reach a stage at which the order (A) is reproduced save that p and q have changed places. This stage has been reached by means of $2m+1$ adjacent interchanges.

Now if, in (2), we change all the signs $2m+1$ times, we end with signs opposite to our initial signs. Accordingly, if the condition (III) of the definition is satisfied for adjacent interchanges of letters, it is automatically satisfied for every interchange of letters.

Second step. The conditions (I), (II), (III) fix the value of the determinant (of the second order)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

to be $a_1 b_2 - a_2 b_1$. For, by (I) and (II), the value must be

$$+a_1 b_2 \pm a_2 b_1,$$

and, by (III), the interchange of a and b must change the signs, so that we cannot have $a_1 b_2 + a_2 b_1$.

Third step. Assume, then, that the conditions (I), (II), (III) are sufficient to fix the value of a determinant of order $n-1$.

By (I), the determinant Δ_n contains a set of terms in which a has the suffix 1; this set of terms is

$$a_1 \sum \pm b_s c_t \ldots k_\theta, \tag{3}$$

wherein, by (I), as applied to Δ_n,

(i) the sum is taken over the $(n-1)!$ possible ways of assigning to $s, t, \ldots, \theta$ the values $2, 3, \ldots, n$ in some order and without repetition.

Moreover, by (II), as applied to Δ_n,

(ii) the term $b_2 c_3 \ldots k_n$ is prefixed by $+$.

Finally, by (III), as applied to Δ_n,

(iii) an interchange of any two of the letters $b, c, \ldots, k$ changes the signs throughout (3).

Hence, by our hypothesis that the conditions (I), (II), (III) fix the value of a determinant of order $n-1$, the terms of (2) in which a has the suffix 1 are given by

$$a_1 \begin{vmatrix} b_2 & c_2 & . & . & k_2 \\ b_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ b_n & c_n & . & . & k_n \end{vmatrix}. \tag{3 a}$$

This, on our assumption that a determinant of order $n-1$ is defined by the conditions (I), (II), (III), fixes the signs of all terms in (2) that contain $a_1 b_2, a_1 b_3, \ldots, a_1 b_n$.

Fourth step. The interchange of a and b in (2) must, by condition (III), change all the signs in (2). Hence the terms of (2) in which b has the suffix 1 are given by

$$-b_1 \begin{vmatrix} a_2 & c_2 & . & . & k_2 \\ a_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ a_n & c_n & . & . & k_n \end{vmatrix}; \tag{3 b}$$

for (3 b) fixes the sign of a term $b_1 a_s c_t \ldots k_\theta$ to be the opposite of the sign of the term $a_1 b_s c_t \ldots k_\theta$ in (3 a).

The adjacent interchanges b with c, c with d, ..., j with k now show that (2) must take the form

$$a_1\begin{vmatrix} b_2 & c_2 & . & . & k_2 \\ b_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ b_n & c_n & . & . & k_n \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & . & . & k_2 \\ a_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ a_n & c_n & . & . & k_n \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & . & . & k_2 \\ a_3 & b_3 & . & . & k_3 \\ . & . & . & . & . \\ a_n & b_n & . & . & k_n \end{vmatrix} -$$

$$-\ldots+(-1)^{n-1}k_1\begin{vmatrix} a_2 & b_2 & . & . & j_2 \\ a_3 & b_3 & . & . & j_3 \\ . & . & . & . & . \\ a_n & b_n & . & . & j_n \end{vmatrix}. \quad (4)$$

That is to say, if conditions (I), (II), (III) define uniquely a determinant of order $n-1$, then they define uniquely a determinant of order n. But they do define uniquely a determinant of order 2, and hence, by induction, they define uniquely a determinant of any order.

2.2. Rule for determining the sign of a given term. If in a term $a_r b_s \ldots k_\theta$ there are λ_n suffixes less than n that come after n, we say that there are λ_n inversions with respect to n. For example, in the term $a_2 b_3 c_4 d_1$, there is one inversion with respect to 4. Similarly, if there are λ_{n-1} suffixes less than $n-1$ that come after $n-1$, we say that there are λ_{n-1} inversions with respect to $n-1$; and so on. The sum

$$N = \lambda_1+\lambda_2+\ldots+\lambda_n$$

is called the total number of inversions of suffixes. Thus, with $n = 6$ and the term

$$a_4 b_3 c_2 d_6 e_1 f_5, \quad (5)$$

$\lambda_6 = 2$, since the suffixes 1 and 5 come after 6,

$\lambda_5 = 0$, since no suffix less than 5 comes after 5,

$\lambda_4 = 3$, since the suffixes 3, 2, 1 come after 4,

$\lambda_3 = 2,\ \lambda_2 = 1,\ \lambda_1 = 0$;

the total number of inversions is $2+3+2+1 = 8$.

If $a_r b_s \ldots k_\theta$ has λ_n inversions with respect to n, then, leaving the order of the suffixes $1, 2,\ldots, n-1$ unchanged, we can make n to be the suffix of the nth letter of the alphabet by λ_n adjacent interchanges of letters and, on restoring alphabetical order, make n the last suffix. For example, in (5), where $\lambda_n = 2$, the two adjacent interchanges f with e and f with d give, in succession,

$$a_4 b_3 c_2 d_6 f_1 e_5, \qquad a_4 b_3 c_2 f_6 d_1 e_5.$$

On restoring alphabetical order in the last form, we have $a_4 b_3 c_2 d_1 e_5 f_6$, in which the suffixes 4, 3, 2, 1, 5 are in their original order, as in (5), and the suffix 6 comes at the end.

Similarly, when n has been made the last suffix, λ_{n-1} adjacent interchanges of letters followed by a restoration of alphabetical order will then make $n-1$ the $(n-1)$th suffix; and so on.

Thus $\lambda_1+\lambda_2+...+\lambda_n$ adjacent interchanges of letters make the term $a_r b_s ... k_\theta$ coincide with $a_1 b_2 ... k_n$. By (III), the sign to be prefixed to any term of (2) is $(-1)^N$, where N, i.e. $\lambda_1+\lambda_2+...+\lambda_n$, is the total number of inversions of suffixes.

2.3. The number N may also be arrived at in another way. Let $1 \leqslant m \leqslant n$. In the term

$$a_r b_s ... k_\theta$$

let there be μ_m suffixes greater than m that come before m. Then the suffix m comes after each of these μ_m greater suffixes and, in evaluating N, accounts for one inversion with respect to each of them. It follows that

$$N = \sum_{m=1}^{n} \mu_m. \tag{6}$$

3. Properties of a determinant

3.1. THEOREM 1. *The determinant Δ_n of § 2.1 can be expanded in either of the forms*

(i) $$\sum (-1)^N a_r b_s ... k_\theta,$$

where N is the total number of inversions in the suffixes $r, s,..., \theta$;

(ii) $$a_1 \begin{vmatrix} b_2 & c_2 & . & . & k_2 \\ b_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ b_n & c_n & . & . & k_n \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & . & . & k_2 \\ a_3 & c_3 & . & . & k_3 \\ . & . & . & . & . \\ a_n & c_n & . & . & k_n \end{vmatrix} + ... +$$

$$+(-1)^{n-1} k_1 \begin{vmatrix} a_2 & b_2 & . & . & j_2 \\ a_3 & b_3 & . & . & j_3 \\ . & . & . & . & . \\ a_n & b_n & . & . & j_n \end{vmatrix}.$$

This theorem has been proved in § 2.

THEOREM 2. *A determinant is unaltered in value when rows and columns are interchanged; that is to say*

$$\begin{vmatrix} a_1 & b_1 & . & . & k_1 \\ a_2 & b_2 & . & . & k_2 \\ . & . & . & . & . \\ a_n & b_n & . & . & k_n \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & . & . & a_n \\ b_1 & b_2 & . & . & b_n \\ . & . & . & . & . \\ k_1 & k_2 & . & . & k_n \end{vmatrix}.$$

By Theorem 1, the second determinant is

$$\sum (-1)^M \alpha_1 \beta_2 \ldots \kappa_n, \tag{7}$$

where α, β,..., κ are the letters a, b,..., k in some order and M is the total number of inversions of letters.

[There are μ_n inversions with respect to k in $\alpha\beta\ldots\kappa$ if there are μ_n letters after k that come before k in the alphabet; and so on: $\mu_1+\mu_2+\ldots+\mu_n$ is the total number of inversions of letters.]

Now consider any one term of (7), say

$$(-1)^M \alpha_1 \beta_2 \ldots \kappa_n. \tag{8}$$

If we write the product with its letters in alphabetical order, we get a term of the form

$$(-1)^M a_r b_s \ldots j_t k_\theta. \tag{9}$$

In (8) there are μ_n letters that come after k, so that in (9) there are μ_n suffixes greater than θ that come before θ. There are μ_{n-1} letters that come before j in the alphabet but after it in (8), so there are μ_{n-1} suffixes greater than t that come before it in (9); and so on. It follows from § 2.3 that M, which is defined as $\sum \mu_n$, is equal to N, where N is the total number of inversions of suffixes in (9).

Thus (7), which is the expansion of the second determinant of the enunciation, may also be written as

$$\sum (-1)^N a_r b_s \ldots k_\theta,$$

which is, by Theorem 1, the expansion of the first determinant of the enunciation; and Theorem 2 is proved.

THEOREM 3. *The interchange of two columns, or of two rows, in a determinant multiplies the value of the determinant by* -1.

It follows at once from (III) of the definition of Δ_n that an interchange of two columns, i.e. an interchange of two letters, multiplies the determinant by -1.

Hence also, by Theorem 2, an interchange of two rows multiplies the determinant by -1.

Corollary. *If a column (row) is moved past an even number of columns (rows), then the value of the determinant is unaltered; in particular*

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 & d_1 \\ c_2 & a_2 & b_2 & d_2 \\ c_3 & a_3 & b_3 & d_3 \\ c_4 & a_4 & b_4 & d_4 \end{vmatrix}.$$

If a column (row) is moved past an odd number of columns (rows), then the value of the determinant is thereby multiplied by -1.

For a column can be moved past an even (odd) number of columns by an even (odd) number of adjacent interchanges. In the particular example, $abcd$ can be changed into $cabd$ by first interchanging b and c, giving $acbd$, and then interchanging a and c.

3.11. The expansion (ii) of Theorem 1 is usually referred to as the expansion by the first row. By the corollary of Theorem 3, there is a similar expansion by any other row. For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and, on expanding the second determinant by its first row, the first determinant is seen to be equal to

$$a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

Similarly, we may show that the first determinant may be written as

$$-a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} + d_2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$