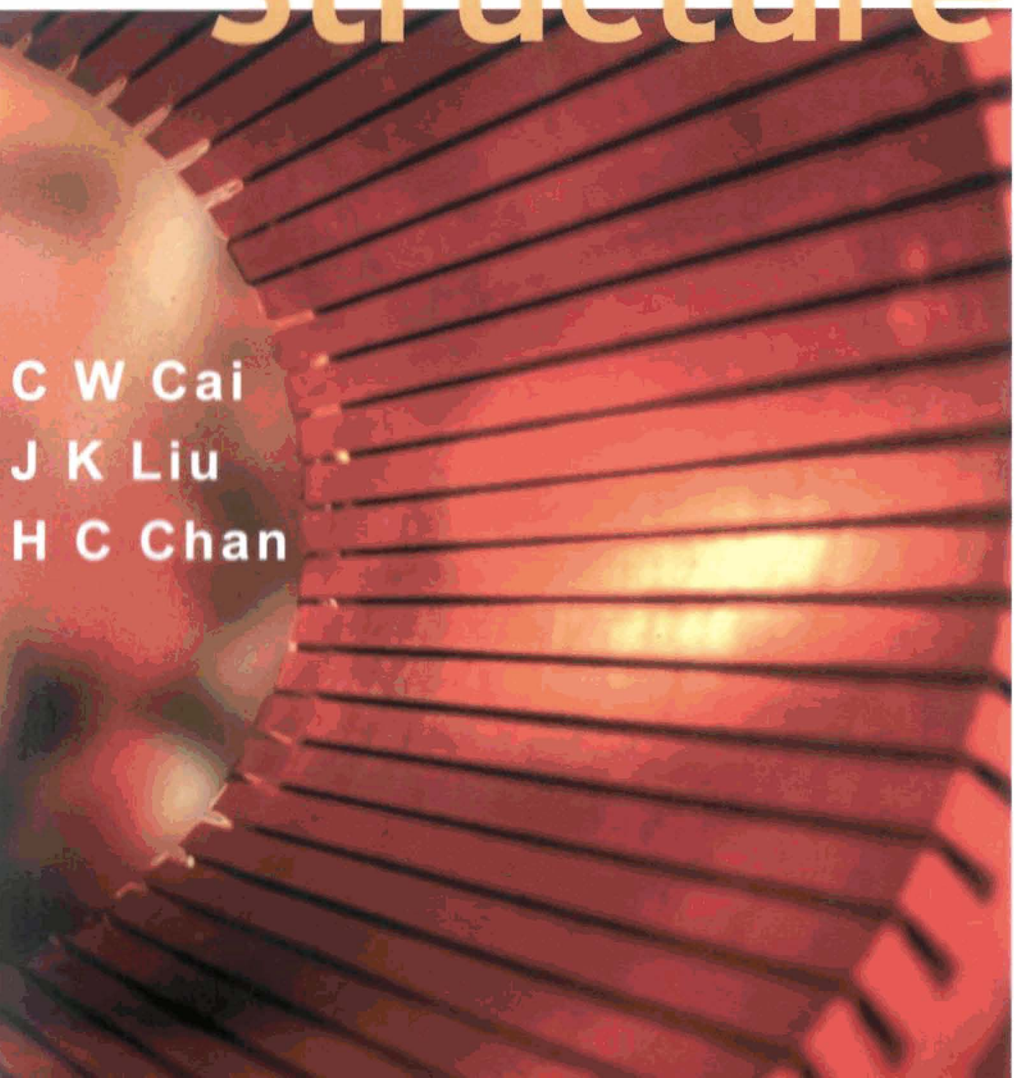
Exact Analysis of Bi-Periodic Structures



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EXACT ANALYSIS OF BI-PERIODIC STRUCTURES

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PREFACE

In the book "Exact Analysis of Structures with Periodicity using U-Transformation" (World Scientific 1998), a comprehensive and systematic explanation has been given on the U-transformation method, its background, physical meaning and mathematical formulation. The book has demonstrated the application of the U-Transformation method in the analyses of many different kinds of periodic structures. As it has been rightly pointed out in the book, the method has a great potential for further development. With the research efforts by the authors and others in recent years, important advancement in the application of the U-transformation method has been made in the following areas:

- The static and dynamic analyses of bi-periodic structures
- Analysis of periodic systems with nonlinear disorder.

The static and dynamic analyses of bi-periodic structures

When the typical substructure in a periodic structure is itself a periodic structure, the original structure is classified as a bi-periodic structure: for example, a continuous truss supported on equidistant supports with multiple equal spans. As a singly periodic structure, the truss within each bay or span between two adjacent supports is a substructure. But there could be many degrees of freedom in such a substructure. If the U-transformation method is applied to analyze this structure as illustrated in the previous book, every uncoupled equation still contains many unknown variables, the number of which is equal to the number of degrees of freedom in each substructure. Therefore, it is not possible to obtain the explicit exact analytical solution yet. Though the substructure is periodic, it is not cyclic periodic. Hence, it is not possible to go any further to apply the same U-transformation technique directly to uncouple the equations. One of the main objectives for writing this new book is to show how to extend the U-transformation technique to uncouple the two sets of unknown variables in a bi-periodic structure to achieve an analytical exact solution. Through an example consisting of a system of masses and springs with bi-periodicity, this book presents a procedure on how to apply the U-transformation technique twice to uncouple the unknowns and get an analytical solution. The book also produces the static and dynamic analyses for certain engineering structures with bi-periodic properties. These include continuous truss with any number of spans, cable network and grillwork on supports with periodicity, and grillwork with periodic stiffening members or equidistant line supports. Explicit exact solutions are given for these examples. The availability of these exact solutions not only helps the checking of the convergence and accuracy of the numerical solutions for these structures, but also provides a basis for the optimization design for these types of structures. It is envisaged that there may be a great prospect for the application of this technique in engineering.

Analysis of periodic systems with nonlinear disorder

The study on the force vibration and localized mode shape of periodic systems with nonlinear disorder is yet another research area that has attained considerable success by the application of the U-transformation method. The localization of the mode shape of nearly periodic systems has been a research topic attracting enormous attention and concern in the past decade. In the same way, localization problem also exists in periodic systems with nonlinear disorder. This book illustrates the analytical approach and procedure for these problems together with the results.

It looks that there are big differences in the physical and mechanical meaning of the problems in the above-mentioned two areas. But as a matter of fact there are similarities in the approaches to their analyses. It is appropriate to present them all together in this book. They are both good examples of the amazing successful application of the U-transformation method.

The advantage of applying the U-transformation method is to make it possible for the linear simultaneous equations, either algebraic or differential equations, with cyclic periodicity to uncouple. The first chapter in this book will provide a rigorous proof for this significant statement and give the form of the uncoupled equations. The result will be used in the procedure to obtain the solutions for the example problems in this book.

Many achievements in this new book are new results that have just appeared in international journals for the first time together with some which have not been published before. This book can be treated as an extension of the previous book "Exact Analysis of Structures with Periodicity using U-Transformation" with the latest advancement and development in the subject. Nevertheless, sufficient details and explanations have been given in this book to make it a new reference book on its own. However, it will be helpful if readers of this book have obtained some ideas of the mathematical procedures and the applications of the U-transformation method from the previous book.

Prof. H.C. Chan

Oct. 30, 2001

CONTENTS

Chapter 1

U TRANSFORMATION AND UNCOUPLING OF GOVERNING EQUATIONS FOR SYSTEMS WITH CYCLIC BI-PERIODICITY

1.1 Dynamic Properties of Structures with Cyclic Periodicity

1.1.1 Governing Equation

In general, the discrete equation for cyclic periodic structures without damping may be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{X}} + \boldsymbol{K}\boldsymbol{X} = \boldsymbol{F} \tag{1.1.1}$$

where a superior dot denotes differentiation with respect to the time variable t, $\boldsymbol{K}$ and $\boldsymbol{M}$ are stiffness and mass matrices and $\boldsymbol{X}$ and $\boldsymbol{F}$ are displacement and loading vectors respectively. Generally they can be written as

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \cdots & \boldsymbol{K}_{1N} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} & \cdots & \boldsymbol{K}_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{K}_{N1} & \boldsymbol{K}_{N2} & \cdots & \boldsymbol{K}_{NN} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \cdots & \boldsymbol{M}_{1N} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \cdots & \boldsymbol{M}_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{M}_{N1} & \boldsymbol{M}_{N2} & \cdots & \boldsymbol{M}_{NN} \end{bmatrix} \tag{1.1.2a, b}$$

and

$$\boldsymbol{X} = \begin{Bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_N \end{Bmatrix}, \qquad \boldsymbol{F} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \vdots \\ \boldsymbol{F}_N \end{Bmatrix} \tag{1.1.3a, b}$$

where N represents the total number of substructures; the vector components $\boldsymbol{x}_j$ and $\boldsymbol{F}_j$ ($j = 1,2,\ldots,N$) denote displacement and loading vectors for the j-th substructure, respectively. The numbers of dimensions of submatrices $\boldsymbol{K}_{rs}$, $\boldsymbol{M}_{rs}$

($r,s=1,2,\ldots,N$) and vector components $\boldsymbol{x}_j$ and $\boldsymbol{F}_j$ ($j=1,2,\ldots,N$) are the same as the degrees of freedom for a single substructure and let J denote the number of degrees of freedom of a substructure.

The stiffness and mass matrices for the cyclic periodic structures possess cyclic periodicity as well as symmetry, namely

$$\boldsymbol{K}_{rs}=\boldsymbol{K}_{sr}^T, \qquad r,s=1,2,\ldots,N \tag{1.1.4}$$

$$\boldsymbol{K}_{11}=\boldsymbol{K}_{22}=\cdots=\boldsymbol{K}_{NN} \tag{1.1.5a}$$

$$\boldsymbol{K}_{1,s}=\boldsymbol{K}_{2,s+1}=\cdots=\boldsymbol{K}_{N-s+1,N}=\boldsymbol{K}_{N-s+2,1}=\cdots=\boldsymbol{K}_{N,s-1}, \quad s=2,3,\ldots,N \tag{1.1.5b}$$

$$\boldsymbol{M}_{rs}=\boldsymbol{M}_{sr}^T, \qquad r,s=1,2,\ldots,N \tag{1.1.6}$$

$$\boldsymbol{M}_{11}=\boldsymbol{M}_{22}=\cdots=\boldsymbol{M}_{NN} \tag{1.1.7a}$$

$$\boldsymbol{M}_{1,s}=\boldsymbol{M}_{2,s+1}=\cdots=\boldsymbol{M}_{N-s+1,N}=\boldsymbol{M}_{N-s+2,1}=\cdots=\boldsymbol{M}_{N,s-1}, \quad s=2,3,\ldots,N \tag{1.1.7b}$$

where $[\ ]^T$ denotes the transposed matrix of $[\ ]$.

The simultaneous equation (1.1.1) with ***K***, ***M*** having cyclic periodicity may be called a cyclic periodic equation.

1.1.2 U Matrix and Cyclic Matrix

Let

$$\boldsymbol{U}=[\boldsymbol{U}_1 \quad \boldsymbol{U}_2 \quad \cdots \quad \boldsymbol{U}_N] \tag{1.1.8a}$$

with the submatrices

$$\boldsymbol{U}_m=\frac{1}{\sqrt{N}}\begin{bmatrix} \boldsymbol{I}_J \\ e^{im\psi}\boldsymbol{I}_J \\ e^{i2m\psi}\boldsymbol{I}_J \\ \vdots \\ e^{i(N-1)m\psi}\boldsymbol{I}_J \end{bmatrix}, \qquad m=1,2,\ldots,N \tag{1.1.8b}$$

in which $\psi = 2\pi/N$, $i = \sqrt{-1}$ and $\boldsymbol{I}_J$ denotes the unit matrix of order *J*.

It can be shown that

$$\overline{\boldsymbol{U}}_r^T \boldsymbol{U}_s = \frac{1}{N}(1 + e^{i(s-r)\psi} + e^{i2(s-r)\psi} + \cdots + e^{i(N-1)(s-r)\psi})\boldsymbol{I}_J$$

$$= \begin{cases} \boldsymbol{I}_J & r = s \\ \dfrac{1}{N}\dfrac{1-e^{iN(s-r)\psi}}{1-e^{i(s-r)\psi}}\boldsymbol{I}_J = 0 & r \neq s \end{cases} \tag{1.1.9}$$

That leads to

$$\overline{\boldsymbol{U}}^T \boldsymbol{U} = \boldsymbol{I} \tag{1.1.10}$$

where the superior bar denotes complex conjugation.

U satisfying Eq. (1.1.10) is referred to as unitary matrix or U matrix. Eq. (1.1.10) indicates that the column vectors of *U* are a set of normalized orthogonal basis in the unitary space with $N \cdot J$ dimensions. The columns of $\boldsymbol{U}_m$ are made up of the basis of the *m*-th subspace with *J* dimensions. An arbitrary vector, say $\boldsymbol{U}_m \boldsymbol{x}_m$ ($\boldsymbol{x}_m$ is a *J* dimensional vector), in the *m*-th subspace possesses the cyclic periodicity.

If

$$\boldsymbol{X}(t) = \boldsymbol{U}_m \boldsymbol{x}_m e^{i\omega t} = \frac{1}{\sqrt{N}} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_m e^{i(\omega t + m\psi)} \\ \boldsymbol{x}_m e^{i(\omega t + 2m\psi)} \\ \vdots \\ \boldsymbol{x}_m e^{i[\omega t + (N-1)m\psi]} \end{Bmatrix} \tag{1.1.11}$$

represents a vibration mode for a cyclic periodic structure with *N* substructures, then this mode is a rotating one, namely the deflection of one substructure has the same amplitude as, and a constant phase difference $m\psi\,(= 2\pi m/N)$ from, the deflection of the preceding substructure. ψ is referred to as the period of the cyclic periodic structure.

All of the rotating modes, the phase difference between two adjacent substructures must be $2\pi m/N$ ($m = 1,2,\ldots,N$) due to cyclic periodicity. As a result, all of the mode vectors lie in the *N* subspaces respectively.

A matrix with cyclic periodicity shown in Eq. (1.1.5) is referred to as cyclic matrix, such as the stiffness and mass matrices of structures with cyclic periodicity are cyclic matrices.

The elementary cyclic matrices can be defined as

$$\boldsymbol{\varepsilon}_j = \begin{array}{c} \begin{matrix} 1 & 2 & \ldots & j & j{+}1 \ldots & N \end{matrix} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & \boldsymbol{I} & & \\ & & & & & \ddots & \\ & & & & & & \boldsymbol{I} \\ \boldsymbol{I} & & & & & & \\ & \ddots & & & & & \\ & & & \boldsymbol{I} & 0 & \cdots & 0 \end{bmatrix} \\ \begin{matrix} 1 & 2 & \ldots & j & \ldots & N \end{matrix} \end{array} \qquad j = 0,1,2,\ldots,N-1 \qquad (1.1.12)$$

where the empty elements are equal to zero, $\boldsymbol{\varepsilon}_0$ is a unit matrix and each element of matrix $\boldsymbol{\varepsilon}_j$ is a J dimensional square matrix. An arbitrary cyclic matrix can be expressed as the series of the elementary cyclic matrix, such as

$$\boldsymbol{K} = \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{K}_{1j} & \\ & & \ddots \end{bmatrix} \boldsymbol{\varepsilon}_{j-1} \quad \text{or} \quad \boldsymbol{K} = \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{K}_{j1} & \\ & & \ddots \end{bmatrix} \boldsymbol{\varepsilon}_{N-j+1} \qquad (1.1.13a)$$

and

$$\boldsymbol{M} = \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{M}_{1j} & \\ & & \ddots \end{bmatrix} \boldsymbol{\varepsilon}_{j-1} \quad \text{or} \quad \boldsymbol{M} = \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{M}_{j1} & \\ & & \ddots \end{bmatrix} \boldsymbol{\varepsilon}_{N-j+1} \qquad (1.1.13b)$$

where $\boldsymbol{\varepsilon}_N \equiv \boldsymbol{\varepsilon}_0$ and $\begin{bmatrix} \ddots & & \\ & \boldsymbol{x} & \\ & & \ddots \end{bmatrix}$ denotes the quasi-diagonal matrix, i.e.,

$$\begin{bmatrix} \ddots & & \\ & \boldsymbol{x} & \\ & & \ddots \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{x} & & & 0 \\ & \boldsymbol{x} & & \\ & & \ddots & \\ 0 & & & \boldsymbol{x} \end{bmatrix} \tag{1.1.14}$$

Noting the cyclic periodicity of $\boldsymbol{U}_m$ and

$$\boldsymbol{\varepsilon}_j \, \boldsymbol{U}_m = e^{ijm\psi} \boldsymbol{U}_m \tag{1.1.15}$$

it can be verified that

$$\overline{\boldsymbol{U}}^T \, \boldsymbol{\varepsilon}_j \boldsymbol{U} = \boldsymbol{\Phi}_j, \qquad j = 0,1,2,\ldots,N-1 \tag{1.1.16a}$$

with

$$\boldsymbol{\Phi}_j \equiv \begin{bmatrix} e^{ij\psi}\boldsymbol{I}_J & & & 0 \\ & e^{i2j\psi}\boldsymbol{I}_J & & \\ & & \ddots & \\ 0 & & & e^{iNj\psi}\boldsymbol{I}_J \end{bmatrix} \tag{1.1.16b}$$

It is obvious that $\boldsymbol{\Phi}_j$ ($j = 0,1,2,\ldots,N-1$) is a diagonal matrix.

One can now apply Eqs. (1.1.13) and (1.1.16) to derive the important formula:

$$\overline{\boldsymbol{U}}^T \boldsymbol{K} \boldsymbol{U} = \sum_{j=1}^{N} \overline{\boldsymbol{U}}^T \begin{bmatrix} \ddots & & \\ & \boldsymbol{K}_{1j} & \\ & & \ddots \end{bmatrix} \boldsymbol{\varepsilon}_{j-1} \boldsymbol{U} = \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{K}_{1j} & \\ & & \ddots \end{bmatrix} \overline{\boldsymbol{U}}^T \, \boldsymbol{\varepsilon}_{j-1} \boldsymbol{U}$$

$$= \sum_{j=1}^{N} \begin{bmatrix} \ddots & & \\ & \boldsymbol{K}_{1j} & \\ & & \ddots \end{bmatrix} \boldsymbol{\Phi}_{j-1} = \begin{bmatrix} \boldsymbol{k}_1 & & & 0 \\ & \boldsymbol{k}_2 & & \\ & & \ddots & \\ 0 & & & \boldsymbol{k}_N \end{bmatrix} \tag{1.1.17}$$

where

$$\boldsymbol{k}_r = \sum_{j=1}^{N} \boldsymbol{K}_{1j} e^{i(j-1)r\psi}, \quad r = 1,2,\ldots,N \tag{1.1.18a}$$

or

$$\boldsymbol{k}_r = \sum_{j=1}^{N} \boldsymbol{K}_{j1} e^{-i(j-1)r\psi}, \quad r = 1,2,\ldots,N \tag{1.1.18b}$$

The matrix $\boldsymbol{k}_r$ is a Hermiltian one, i.e.,

$$\bar{\boldsymbol{k}}_r^T = \boldsymbol{k}_r \tag{1.1.19}$$

In the same way, we have

$$\bar{\boldsymbol{U}}^T \boldsymbol{M} \boldsymbol{U} = \begin{bmatrix} \boldsymbol{m}_1 & & & 0 \\ & \boldsymbol{m}_2 & & \\ & & \ddots & \\ 0 & & & \boldsymbol{m}_N \end{bmatrix} \tag{1.1.20}$$

and

$$\boldsymbol{m}_r = \sum_{j=1}^{N} \boldsymbol{M}_{1j} e^{i(j-1)r\psi}, \quad r = 1,2,\ldots,N \tag{1.1.21a}$$

or

$$\boldsymbol{m}_r = \sum_{j=1}^{N} \boldsymbol{M}_{j1} e^{-i(j-1)r\psi}, \quad r = 1,2,\ldots,N \tag{1.1.21b}$$

$\boldsymbol{m}_r$ is also a Hermiltian matrix.

1.1.3 U Transformation and Uncoupling of Simultaneous Equations with Cyclic Periodicity

The U transformation can be defined as

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{q} \tag{1.1.22}$$

where $\boldsymbol{X}, \boldsymbol{U}$ are defined as Eqs. (1.1.3a) and (1.1.8) respectively and

$$\boldsymbol{q} \equiv \begin{Bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \vdots \\ \boldsymbol{q}_N \end{Bmatrix} \tag{1.1.23}$$

and $\boldsymbol{q}_m$ ($m = 1,2,\ldots,N$) are vectors of dimension J.

Because the coefficient matrix in Eq. (1.1.22) is an unitary matrix satisfying Eq. (1.1.10), the complex linear transformation (1.1.22) is referred to as U transformation. Recalling Eq. (1.1.10), premultiplying both sides of Eq. (1.1.22) by $\overline{\boldsymbol{U}}^T$, the inverse U transformation can be obtained as

$$\boldsymbol{q} = \overline{\boldsymbol{U}}^T \boldsymbol{X} \tag{1.1.24}$$

The component forms of the U and inverse U transformations are

$$\boldsymbol{x}_j = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{i(j-1)m\psi} \boldsymbol{q}_m, \qquad j = 1,2,\ldots,N \tag{1.1.25a}$$

and

$$\boldsymbol{q}_m = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i(j-1)m\psi} \boldsymbol{x}_j, \qquad m = 1,2,\ldots,N \tag{1.1.25b}$$

with $\psi = 2\pi/N$ and $i = \sqrt{-1}$.

Usually, the original variables $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ are real vectors representing the displacement vectors of substructures for a cyclic periodic structure and $\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_N$ are a set of generalized displacement vectors. Noting Eq. (1.1.25b) and $\boldsymbol{x}_j$ being real vector, it can be shown that

$$\boldsymbol{q}_{N-m} = \overline{\boldsymbol{q}}_m \tag{1.1.26a}$$

and

$$q_N,\quad q_{N/2} \text{ (if N is even)} = \text{real vector} \tag{1.1.26b}$$

Applying the U transformation (1.1.22) to Eq. (1.1.1), namely substituting Eq. (1.1.22) into Eq. (1.1.1) and premultiplying both sides of Eq. (1.1.1) by $\overline{U}^T$, we have

$$\begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_N \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_N \end{Bmatrix} + \begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_N \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix} \tag{1.1.27}$$

where

$$f_r = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i(j-1)r\psi} F_j \tag{1.1.28}$$

Eq. (1.1.27) is made up of N independent equations, i.e.,

$$m_r \ddot{q}_r + k_r q_r = f_r, \quad r = 1,2,\ldots,N \tag{1.1.29}$$

Noting the definitions of m_r, k_r and f_r shown in Eqs. (1.1.21), (1.1.18) and (1.1.28) respectively, it is obvious that

$$m_{N-r} = \overline{m}_r, \quad k_{N-r} = \overline{k}_r, \quad f_{N-r} = \overline{f}_r \tag{1.1.30}$$

and m_N, $m_{N/2}$ (if N is even), k_N, $k_{N/2}$ (if N is even) are real symmetric matrices, so $q_{N-r} = \overline{q}_r$ and q_N, $q_{N/2}$ (if N is even) are real vectors.

We need only consider $\frac{N}{2}+1$ (N is even) or $\frac{N+1}{2}$ (N is odd) equations, i.e., $r = 1,2,\ldots,\frac{N}{2},N$ (N is even) or $r = 1,2,\ldots,\frac{N-1}{2},N$ (N is odd) in Eq. (1.1.29).

1.1.4 Dynamic Properties of Cyclic Periodic Structures

Consider now the natural vibration of rotationally periodic structures. The natural vibration equation can be expressed in terms of the generalized displacements as

$$\boldsymbol{k}_r \boldsymbol{q}_r = \omega^2 \boldsymbol{m}_r \boldsymbol{q}_r, \quad r = 1,2,\ldots,N \tag{1.1.31}$$

where ω denotes the natural frequency, $\boldsymbol{q}_r$ represents the amplitude of the r-th generalized displacement and $\boldsymbol{k}_r$, $\boldsymbol{m}_r$ denote generalized stiffness and mass matrices as shown in Eqs. (1.1.18) and (1.1.21) respectively.

It is well known that the eigenvalues of the eigenvalue equation (1.1.31) with Hermiltian matrices are real numbers. The eigenvalues can be denoted as $\omega_{r,1}^2$, $\omega_{r,2}^2,\cdots$, $\omega_{r,J}^2$ ($\omega_{r,s}^2 \le \omega_{r,s+1}^2, s = 1,2,\ldots,J-1$) and the corresponding normalized orthogonal eigenvectors may be written as $\boldsymbol{q}_{r,1}$, $\boldsymbol{q}_{r,2}$, ..., $\boldsymbol{q}_{r,J}$. They satisfy the eigenvalue equation and the normalized orthogonal condition, i.e.,

$$\boldsymbol{k}_r \boldsymbol{q}_{r,s} = \omega_{r,s}^2 \boldsymbol{m}_r \boldsymbol{q}_{r,s}, \quad s = 1,2,\ldots,J;\ r = 1,2,\ldots,N \tag{1.1.32}$$

and

$$\bar{\boldsymbol{q}}_{r,s}^T \boldsymbol{m}_r \boldsymbol{q}_{r,s} = 1, \quad s = 1,2,\ldots,J;\ r = 1,2,\ldots,N \tag{1.1.33}$$

leading to

$$\omega_{r,s}^2 = \bar{\boldsymbol{q}}_{r,s}^T \boldsymbol{k}_r \boldsymbol{q}_{r,s} = \text{real number} \tag{1.1.34}$$

Noting $\boldsymbol{k}_{N-r} = \bar{\boldsymbol{k}}_r$, $\boldsymbol{m}_{N-r} = \bar{\boldsymbol{m}}_r$, it is obvious that

$$\omega_{N-r,s}^2 = \omega_{r,s}^2, \quad \boldsymbol{q}_{N-r,s} = \bar{\boldsymbol{q}}_{r,s}, \quad s = 1,2,\ldots,J \tag{1.1.35}$$

and $\boldsymbol{q}_{N,s}$, $\boldsymbol{q}_{\frac{N}{2},s}$ (if N is even) $s = 1,2,\ldots,J$ are real eigenvectors.

Let us consider the natural modes. Corresponding to the eigenvector $\boldsymbol{q}_{N,s}$ ($s = 1,2,\ldots,J$), the natural mode can be expressed as

$$X = U_N q_{N,s} \tag{1.1.36a}$$

leading to

$$x_1 = x_2 = x_3 = \cdots = x_N = \frac{1}{\sqrt{N}} q_{N,s} \tag{1.1.36b}$$

The vibrating displacements of all substructures possess the same amplitude vector and vibrating phase.

Corresponding to the eigenvector $q_{\frac{N}{2},s}$ (if N is even), the natural mode is

$$X = U_{\frac{N}{2}} q_{\frac{N}{2},s} \tag{1.1.37a}$$

which leads to

$$x_1 = -x_2 = x_3 = \cdots = -x_N = \frac{1}{\sqrt{N}} q_{\frac{N}{2},s} \tag{1.1.37b}$$

so the displacement vectors for any two adjacent substructures are equal and opposite. Such a mode doesn't occur when N is odd.

For other natural frequencies $\omega_{r,s}$ $(r \neq N, \frac{N}{2}; s = 1,2,\ldots,J)$, which are repeated frequencies: $\omega_{r,s} = \omega_{N-r,s}$ $(r \neq N, \frac{N}{2}; s = 1,2,\ldots,J)$, the corresponding modes take the form as

$$X = U_r q_{r,s} \tag{1.1.38a}$$

or

$$x_k = \frac{1}{\sqrt{N}} e^{i(k-1)r\psi} q_{r,s}, \quad k = 1,2,\ldots,N \tag{1.1.38b}$$

This is a rotating mode, the deflection of any substructure has a constant phase difference $2\pi r/N$ from that for the preceding substructure, i.e., the mode lies in the r-th mode subspace. The real and imaginary parts of each rotating mode are two independent standing modes corresponding to the same natural frequency.