SET THEORY

ON THE STRUCTURE OF THE REAL LINE

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Dedicated to Saharon Shelah

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Preface

This book reflects the current progress in an important segment of descriptive set theory. Its main focus is measure and category in set theory, most notably asymmetry results.

The book consists of three interconnected parts: results that can be proven in ZFC and its extensions, independence results, and the "tools" used in investigating the first two parts. Most chapters can be read independently of the others – the exception is chapter 7 that relies heavily on the material from the previous chapter. While we have included some background material concerning forcing, forcing iteration, and the theory of projective sets, we assume that the reader is comfortable with basic set theoretical techniques. In particular, we take for granted the reader's familiarity with [Kunen, 1980] and parts I, II and IV of [Jech, 1978].

Most of the material presented in this book was developed during last fifteen years. Hopefully, we have assigned authorship to the theorems appropriately. One consequence of the rapid development of this subject is that some results have lost their reference. We apologize to those who feel that their results are unreferenced.

We wish to express our thanks to those who helped us in the preparation of our text. Miro Repický proofread chapters 4, 6 and 9 – the final version of chapter 6 is based on his paper [Repický, 1994] and on his correspondence [Repický, 1992]; Andrzej Rosłanowski proofread chapter 3; Janusz Pawlikowski and Jorg Brendle read chapter 2, and Martin Goldstern read chapter 1 and 6. They made *many* corrections, found simpler proofs, and offered other positive suggestions. We also want to thank David Fremlin, Lorenz Halbeisen, Vladimir Kanovei, Jose Ruiz, Tzvi Scarr, Marion Scheepers, and Boaz Tsaban for many helpful remarks and corrections.

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Boise, May 1995 Tomek Bartoszyński

CHAPTER 1

Introduction

In this chapter we have gathered some of the pertinent notation as well as some required background material. For further information see [Jech, 1978], [Jech, 1986] and [Kunen, 1980].

1.1 Logic and Combinatorics

1.1.A Set theory The set theory we will use is the Zermelo-Fraenkel, ZFC. Quite often we will use models of "sufficiently large" fragments of ZFC. We will reserve the collective name $\mathsf{ZFC}^\star$ for these fragments and we will never investigate the precise definition of "sufficiently large."

We will also use the standard set theoretic notation. In particular, $X \setminus Y$ denotes the difference between sets X, and Y and $X \triangle Y = (Y \setminus X) \cup (X \setminus Y)$. Furthermore,

$$\mathcal{P}(X) = \{Y : Y \subseteq X\}.$$

By the term "model" we will mean a set together with the relation $\in$. For models N, M we say that N is an elementary submodel of M, $N \prec M$, if for every formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in N$, we have

$$N \models \varphi(a_1, \ldots, a_n) \iff M \models \varphi(a_1, \ldots, a_n).$$

We will define the cumulative hierarchy as follows: $\mathbf{V}_0 = \emptyset$, $\mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V}_\alpha)$, and $\mathbf{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha$ for a limit ordinal λ. $\mathbf{V} = \bigcup_{\alpha \in \mathsf{Ord}} \mathbf{V}_\alpha$ is a standard model of ZFC, where Ord denotes the class of all ordinals. For $x \in \mathbf{V}$ let $\mathsf{rank}(x) = \min\{\alpha : x \in \mathbf{V}_\alpha\}$.

Constructible universe $\mathbf{L}$ is the canonical inner model of ZFC, and as above $\mathbf{L} = \bigcup_{\alpha \in \mathsf{Ord}} \mathbf{L}_\alpha$, where $\mathbf{L}_{\alpha+1}$ consists of subsets of $\mathbf{L}_\alpha$ definable in $\mathbf{L}_\alpha$, while $\mathbf{L}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha$ for a limit ordinal λ. Similarly, if $X \subseteq \mathsf{Ord}$ is a set we define the model $\mathbf{L}[X]$. The case when $X \subseteq \omega$ will be most important for us. We refer the reader to [Kunen, 1980] and [Jech, 1978] for the details.

One of the models of $\mathsf{ZFC}^\star$ that we will use quite often is $\mathbf{H}(\kappa)$, the collection of sets that are hereditarily of size $< \kappa$. We will work with $\mathbf{H}(\aleph_0)$ or $\mathbf{H}(\aleph_1)$ or with $\mathbf{H}(\chi)$, where χ is a "sufficiently large" regular cardinal.

The quantifiers $\forall^\infty$ and $\exists^\infty$ will denote "for all but finitely many" and "for infinitely many," respectively.

1.1.B Sets and functions For a set X let $|X|$ denote the cardinality of X. For sets X, Y let $X \times Y$ denote the cartesian product of X and Y. If $A \subseteq X \times Y$, let

$$(A)_x = \{y \in Y : \langle x, y\rangle \in A\} \text{ for } x \in X, \text{and}$$

$$(A)^y = \{x \in X : \langle x, y\rangle \in A\} \text{ for } y \in Y.$$

f is a function, if $f \subseteq \mathsf{dom}(f) \times \mathsf{range}(f)$, where $\mathsf{dom}(f)$ and $\mathsf{range}(f)$ denote the domain and range of f, respectively, and $|(f)_x| = 1$. We will always use the conventional $f(x)$ instead of $(f)_x$.

For $X \subseteq \mathsf{dom}(f)$ and $Y \subseteq \mathsf{range}(f)$, let

$$f(X) = \{f(x) : x \in X\}$$

and

$$f^{-1}(Y) = \{x : f(x) \in Y\}$$

denote the image of X and the preimage of Y, respectively. In those few cases where the image of X gets confused with the value of f at X, we will be more explicit.

If $Z \subseteq \mathsf{dom}(f)$, then $f{\restriction}Z$ denotes the restriction of f to Z. For functions f, g let $f \circ g$ denote the composition of f and g.

When X is a set and κ is a cardinal, let X^κ denote the set of all sequences of elements of X of length κ, and let $[X]^\kappa$ denote the set of all subsets of X of size κ. Similarly let $X^{<\kappa} = \bigcup_{\xi<\kappa} X^\xi$ and $[X]^{<\kappa} = \bigcup_{\xi<\kappa}[X]^\xi$.

In particular, the sets $X^{<\omega}$ and $[X]^{<\omega}$ are the sets of finite sequences of elements of X and finite subsets of X, respectively. If s is a sequence, then $|s|$ is the length of s. For $s, t \in X^{<\omega}$ let $s^\frown t$ to be the concatenation of s and t, and $[s] = \{f \in X^\omega : s \subseteq f\}$.

The following theorem is known as the Δ-lemma.

Theorem 1.1.1 ([Kunen, 1980])
Suppose that $\mathcal{A}$ is an uncountable family of finite sets. Then there exists an uncountable subfamily $\mathcal{A}' \subseteq \mathcal{A}$ and a set A_0 such that for all $A, B \in \mathcal{A}'$, $A \cap B = A_0$. □

For a cardinal number κ, let κ^+ denote the cardinal successor of κ. Let GCH denote the statement "$\forall\kappa\ 2^\kappa = \kappa^+$," and let CH denote the statement "$2^{\aleph_0} = \aleph_1$".

1.1.C Stationary sets Suppose that κ is an infinite ordinal and $X \subseteq \kappa$.

1. X is closed if for every $\alpha < \kappa$, $\sup(X \cap \alpha) \in X$.

2. X is unbounded if $X \setminus \alpha \neq \emptyset$ for all $\alpha < \kappa$.

LEMMA 1.1.2 ([JECH, 1978]) *Suppose that κ is a regular uncountable cardinal and $\{X_\alpha : \alpha < \lambda < \kappa\}$ are closed unbounded sets. Then $X = \bigcap_{\alpha<\lambda} X_\alpha$ is a closed unbounded set.* □

In other words, closed unbounded sets generate a κ-complete filter on κ.

Suppose that $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ. We define the diagonal intersection of the X_α's as:

$$\triangle_{\alpha<\kappa} X_\alpha = \left\{ \delta < \kappa : \delta \in \bigcap_{\alpha<\delta} X_\alpha \right\}.$$

LEMMA 1.1.3 ([JECH, 1978]) *The diagonal intersection of a κ-sequence of closed unbounded subsets of κ is closed and unbounded.* □

A set $X \subseteq \kappa$ is stationary if $X \cap C \neq \emptyset$ for every closed unbounded set $C \subseteq \kappa$.

Suppose that $S \subseteq \kappa$. A function $f : S \longrightarrow \mathsf{Ord}$ is called regressive if $f(\alpha) < \alpha$ for all $\alpha \in X \setminus \{0\}$.

THEOREM 1.1.4 (FODOR, [JECH, 1978])
If f is a regressive function on a stationary set $X \subseteq \kappa$, then there exists a stationary set $Y \subseteq X$ and $\delta < \kappa$ such that $f(\alpha) = \delta$ for all $\alpha \in Y$. □

THEOREM 1.1.5 ([JECH, 1978])
Suppose that κ is a regular cardinal and $X \subseteq \kappa$ is a stationary set. Then there exists a partition $\{X_\alpha : \alpha < \kappa\}$ of X such that X_α is stationary for all $\alpha < \kappa$. □

1.1.D Trees A partial ordering $(T, <)$ is called a tree if for every $t \in T$ the set $\{s \in T : s < t\}$ is well–ordered.

The α-th level of T, which we denote by T_α, is the set of all $t \in T$ such that the order type of $\{s \in T : s < t\}$ is α. For a tree T and $t \in T$ let $\mathsf{succ}_T(t)$ be the set of all immediate successors of t in T, and let $T_t = \{s \in T : s < t \text{ or } t < s\}$ be the subtree determined by t. A branch through T is any maximal linearly ordered subset of T. The set of all branches of T is denoted by $[T]$. An antichain is any set of pairwise incompatible elements of T.

Let $\mathsf{split}(T) = \{t \in T : |\mathsf{succ}_T(t)| > 1\}$ and let $\mathsf{stem}(T)$ be the first element of $\mathsf{split}(T)$. A tree T is called perfect if for every $s \in T$ there exists $t \in \mathsf{split}(T)$ such that $s < t$.

The most common examples of trees in this text are those trees on ω that are subtrees of $(\omega^{<\omega}, \subseteq)$. We will identify an infinite branch b in $\omega^{<\omega}$ with the function $\bigcup b \in \omega^\omega$.

DEFINITION 1.1.6
Suppose that κ is an infinite cardinal and T is a tree of size κ. We say that

1. *T is a κ-Aronszajn tree if all levels of T have size $< \kappa$, and T has no branches of size κ, and*

2. *T is a κ-Suslin tree if all antichains and branches of T have size $< \kappa$.*

When $\kappa = \aleph_1$ we will call these trees Aronszajn and Suslin trees, respectively.

We have the following classical result:

THEOREM 1.1.7 (KÖNIG)
If T is an infinite tree such that $\mathsf{succ}_T(t)$ is finite for all $t \in T$, then T has an infinite branch. In particular, there are no $\aleph_0$-Aronszajn trees. □

In 3.3.6 on page 140, we will show that an Aronszajn tree can be constructed in ZFC. On the other hand, a Suslin tree cannot be constructed in ZFC, but it can be built in the theory $\mathsf{ZFC} + \mathbf{V} = \mathbf{L}$ (see [Jech, 1978]).

A tree T is called special if $T = \bigcup_{n \in \omega} A_n$, where each A_n is an antichain in T.

THEOREM 1.1.8 ([JECH, 1978])
$\mathbf{MA}_{\aleph_1}$ implies that every Aronszajn tree is special. In particular, $\mathbf{MA}_{\aleph_1}$ implies that there are no Suslin trees. □

1.1.E Large cardinals We will use several definitions associated with large cardinals.

DEFINITION 1.1.9
Suppose that κ is a regular uncountable cardinal.

1. *κ is inaccessible if $2^\lambda < \kappa$ for all $\lambda < \kappa$.*

2. *κ is Mahlo if $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary.*

3. *κ is weakly compact if there are no κ-Aronszajn trees.*

We will often use the following classical lemma:

LEMMA 1.1.10 ([JECH, 1978]) *The following conditions are equivalent:*

1. *$\aleph_1^{\mathbf{L}[a]} < \aleph_1$ for all $a \in \mathbb{R}$, and*

2. *$\aleph_1$ is an inaccessible cardinal in $\mathbf{L}[a]$ for all $a \in \mathbb{R}$.* □

We will use the following characterization of weakly compact cardinals.

DEFINITION 1.1.11
A cardinal κ is $\mathbf{\Pi}^1_1$-indescribable if for every $\mathbf{\Pi}^1_1$ sentence φ and every set $U \subseteq \kappa$,

$$(\mathbf{V}_\kappa, \in, U) \models \varphi \iff \exists \lambda < \kappa \ (\mathbf{V}_\lambda, \in, U \cap \lambda) \models \varphi.$$

THEOREM 1.1.12 ([JECH, 1978])
A cardinal κ is weakly compact if, and only if, it is $\mathbf{\Pi}^1_1$-indescribable. □

1.2 Topology and Measure

The subject of this book is sets of real numbers but for technical reasons we will often work in other topological spaces. Below we define several spaces whose elements we will identify with "reals." These spaces are the real line $\mathbb{R}$, the unit interval $[0, 1]$, the Cantor set 2^ω, the Baire space ω^ω, and the space $\mathcal{X}_f = \prod_{n\in\omega} f(n)$, where $f \in \omega^\omega$.

Recall that X is a Polish space if X is homeomorphic to a complete metric space with no isolated points. Most of the results in this book can be formulated and proved for Polish spaces equipped with a measure.

Let (X, d) be a Polish space with a metric d. For a set $A \subseteq X$ let

$$\mathsf{cl}(A) = \{y \in X : \forall \varepsilon > 0 \ \exists x \in A \ d(x, y) < \varepsilon\}$$

be the closure of A in X.

1.2.A Borel sets Let X be a Polish space. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a σ-algebra if

1. $\emptyset, X \in \mathcal{A}$,
2. if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$, and
3. if $\{A_n : n \in \omega\} \subseteq \mathcal{A}$, then $\bigcup_{n\in\omega} A_n \in \mathcal{A}$.

Let $\mathsf{BOREL}(X)$ be the smallest σ-algebra containing all open subsets of X.

It is useful to give a more explicit definition. Let $\mathbf{\Sigma}^0_1$ be the collection of all open subsets of X and $\mathbf{\Pi}^0_1$ be the collection of all closed subsets of X.

For $\alpha > 1$, let

$$\mathbf{\Sigma}^0_\alpha = \left\{\bigcup_{n\in\omega} A_n : \forall n \ \exists \beta < \alpha \ A_n \in \mathbf{\Pi}^0_\beta\right\} \text{ and } \mathbf{\Pi}^0_\alpha = \left\{X \setminus A : A \in \mathbf{\Sigma}^0_\alpha\right\}.$$

Traditionally $\mathbf{\Sigma}^0_2$ sets are called F_σ sets and $\mathbf{\Pi}^0_2$ sets are called G_δ sets. Using the axiom of choice we easily show that

$$\mathsf{BOREL}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha.$$

We say that $A \subseteq X$ is nowhere dense if $\mathsf{cl}(X \setminus \mathsf{cl}(A)) = X$. In other words, A is nowhere dense if its closure does not contain an open set.

A set $F \subseteq X$ is meager if $F = \bigcup_{n\in\omega} F_n$ where each set F_n is nowhere dense. Meager sets are also called first category sets.

A set $A \subseteq X$ has the Baire property if there exists a Borel set B such that the symmetric difference $A \triangle B$ is meager. The collection of all subsets of X having the Baire property will be denoted by $\mathsf{BAIRE}(X)$. We have the following classical result:

THEOREM 1.2.1 (BAIRE)
Let X be a Polish space. Then X is not meager. □

From the above definitions it follows that every nowhere dense set can be covered by a closed nowhere dense set and that every meager set can be covered by a meager set of type F_σ. We will always deal with the set in this form. In particular, meager will always mean "meager of type F_σ," and nowhere dense will mean "closed nowhere dense." Moreover, we have the following:

THEOREM 1.2.2 ([JECH, 1978])
Suppose that A has the Baire property. Then there exists an open set U such that $A \triangle U$ is meager. □

The convenience of working in the Baire space or the Cantor set instead of an arbitrary Polish space stems from the fact that many topological notions in those spaces have combinatorial counterparts. In particular, $\{[s] : s \in \omega^{<\omega}\}$ is a basis in the Baire space and $\{[s] : s \in 2^{<\omega}\}$ is a basis in 2^ω.

Suppose that $C \subseteq 2^\omega$ is a set. Let $T = \{x{\restriction}n : n \in \omega,\ x \in C\}$. Clearly, T is a subtree of $2^{<\omega}$, and we easily see that $\mathsf{cl}(C) = [T]$. In particular, closed subsets of 2^ω correspond to trees in $2^{<\omega}$.

In ω^ω we can show even more:

LEMMA 1.2.3 *Let $K \subseteq \omega^\omega$ be a closed set. The following conditions are equivalent.*

1. *K is compact,*
2. *$K = [T]$, where T is a tree such that $\mathsf{succ}_T(t)$ is finite for every $t \in T$, and*
3. *there exists $f \in \omega^\omega$ such that $K \subseteq \{x \in \omega^\omega : \forall n\ x(n) < f(n)\}$.*

PROOF. (1) $\to$ (2) Suppose that $K = [T]$. If for some $n \in \omega$, T_n is infinite, then $\{[s] : s \in T_n\}$ is an open cover of K with no finite subcovering and contradicts (1).

The remaining implications are obvious. □

A set $A \subseteq 2^\omega$ is called clopen if A is both closed and open. Note that 2^ω, and in general any compact metric space, has only countably many clopen subsets.

A closed set $A \subseteq 2^\omega$ is called perfect if $A = [T]$, where T is a perfect tree.

We have the following classical result:

THEOREM 1.2.4 (CANTOR-BENDIXON, [KECHRIS, 1995])
Let $A \subseteq 2^\omega$ be a closed set. Then $A = P \cup X$, where P is perfect and X is countable. □

1.2.B Group structure For $x, y \in [0,1]$, by $x + y$ we mean $x + y$, if $x + y \leq 1$ and $x + y - 1$ otherwise. Similarly for $x, y \in 2^\omega$, $(x + y)$ is defined as $(x + y)(n) = x(n) + y(n) \mod 2$. Similarly, for $h, g \in \mathcal{X}_f$, $(h + g)(n) = h(n) + g(n) \mod f(n)$.

The operation of addition defined above gives the underlying space a structure of the topological group.

Suppose that X is a Polish space. Throughout this book we will refer to a function $\mu : \mathsf{BOREL}(X) \longrightarrow [0,1]$ as measure if it has the following properties:

1. $\mu(\emptyset) = 0$, $\mu(X) = 1$,
2. if $\{A_n : n \in \omega\} \subseteq \mathsf{BOREL}(X)$ is a sequence of pairwise disjoint sets then $\mu\left(\bigcup_{n\in\omega} A_n\right) = \sum_{n\in\omega} \mu(A_n)$,
3. μ is nonatomic, that is, for every set $A \in \mathsf{BOREL}(X)$, such that $\mu(A) > 0$, there exists $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$,
4. for $A \in \mathsf{BOREL}(X)$ and $t \in X$, $\mu(t + A) = \mu(A)$ (if X is a topological group), and
5. for every $A \in \mathsf{BOREL}(X)$ and $\varepsilon > 0$, there exists a compact set K and an open set U such that $K \subseteq A \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

Note that, when μ is a measure, it can be extended to a σ-algebra

$$\mathsf{MEASURABLE}(X) = \{A : \exists B \in \mathsf{BOREL}(X)\ \mu(A \triangle B) = 0\}.$$

We will call the elements of $\mathsf{MEASURABLE}(X)$, measurable sets.

We will use the concept of outer and inner measure. For any set $A \subseteq X$, define the outer measure of A as

$$\mu^\star(A) = \inf\{\mu(U) : A \subseteq U \ \&\ U \text{ is open}\}$$