INTRODUCTION TO

The Theory of Functions of A Complex Variable

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To

H. J. W. ✠

and

A. J. T.

PREFACE

A student of mathematics who is taking a course in the theory of functions of a complex variable should in general be mature enough to appreciate a rigorous treatment of the subject. The better student will even have become quite impatient with the phrase "it can be shown" which he is likely to have encountered only too frequently in the years before. A definite need thus exists for an introduction to functions of complex variables in which all results are derived from a simple set of axioms. Since no such book is available I have attempted to fill the gap by writing a text in which occur neither "intuitive proofs" nor theorems for whose proofs the reader is referred to other sources. The only intentional exception to this rule is the omission of those proofs that seemed to be simple enough to be left as exercises for the reader. In certain borderline cases outlines of proofs are given. Thus the statement that no previous mathematical knowledge is required of the reader is literally true.

It is hoped that this book will be useful as a text. I have used preliminary drafts of it in a two-semester course at Washington University in the academic years 1946–47 and 1948–49. It also is intended to serve as a reference book to which instructors and interested students may want to turn for complete derivations in many instances not to be found elsewhere.

Intuition and rigor are fundamental in mathematics. That the former appears only in occasional remarks in the following pages should not be taken to mean that I underrate its importance. However intuition is usually overemphasized in the early years of a student's mathematical training, and it was part of my job, as I saw it, to rectify this. In addition by its very nature intuition is subjective and as such is best developed by the student himself to suit his own needs.

The task I set myself demanded development of the basic tools of analysis and the inclusion of a proof of Jordan's curve theorem together with a careful treatment of other geometric concepts. In defining a positively directed curve and an angle I have made use of integrals and trigonometric functions, respectively. This might be unsatisfactory in a purely geometrical exposition, but it seemed justified since space could thus be saved for other topics. The same consideration, namely, that I wanted to cover as much material as possible within a limited space, has led me to deviate from the standard methods of proof in some other cases also. Since it did not seem to be in keeping with the purpose of this book, I have made no attempt to determine where credit for any particular theorem or proof may lie. Accordingly I do not claim that my treatment contains any new results.

Though such topics as Cauchy's integral theorem, power series, analytic continuation, Riemann's mapping theorem constitute in my opinion essential elements of any introduction to complex variable theory, the choice of other topics as well as the order of presentation was essentially a matter of taste. I relegated the Cauchy-Riemann conditions to a chapter near the end of the text and base the early development of the theory solely on the Cauchy integral theorem. In this way only existence of $f'(z)$ has to be assumed (instead of continuity of $f'(z)$) and $f(z)$ can be considered as one function of a single complex variable (instead of as two real-valued functions of two real variables). The inclusion of the Stieltjes-Vitali theorem was at least in part motivated by the opportunity thus presented to discuss function spaces and thereby tie-in function theory with other branches of mathematics. As far as space permitted I have tried in other places as well to point out connections with more abstract disciplines.

The reading of Landau's books has been an important factor in developing my preference for the "Satz-Beweis" style of mathematical writing. The proof of the Jordan curve theorem as well as the development of the necessary tools (Ch. 12 and beginning of Ch. 13) is essentially the one given by Newman in his *Elements of the Topology of Plane Sets of Points.* I have also consulted the works of Bieberbach, Hurwitz-Courant, Knopp, Pringsheim, and Titchmarsh.

Finally I would like to express my thanks to Professor Walter Leighton for encouraging me to start this project and to carry it to completion and to Frl. Vogt of the Mathematische Institut of the Freie Universität Berlin for typing the manuscript.

Saint Louis, January 1953

W. J. Thron

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FUNDAMENTAL CONCEPTS

REMARK 1.1. Mathematics is so interrelated with language and logic that a rigorous development of it from the very beginning would necessitate a simultaneous development of language and logic. Because this would carry us much too far afield we make a compromise. We shall assume that the reader is familiar with the English language and with the language and rules of logic. We shall, however, attempt to bring out clearly the axioms and definitions of mathematics. Typical of the difficulties arising out of this compromise is the following: We shall assume that the reader is familiar with the term "class," which is a term of logic, and we shall use this term to describe some of the properties we require the mathematical objects, called sets, to have.

AXIOM 1.1. We postulate the existence of *objects*. Objects can be *equal* or *unequal* to each other (notation: $a = b$ or $a \neq b$). The equality relation is required to satisfy the following properties:

(i) Given any objects a and b either $a = b$ or $a \neq b$, but the two statements cannot be true simultaneously.
(ii) For every object a, $a = a$.
(iii) If $a = b$, then $b = a$.
(iv) If $a = b$ and $b = c$, then $a = c$.

AXIOM 1.2. There exist objects 0, 1 with $0 \neq 1$.

AXIOM 1.3. There exist special objects, called *sets*. Objects and sets are related by the relation to be *an element of* or to be *not an element of* (notation: $a \in A$ or $a \notin A$). This relation is required to satisfy the following conditions:

(i) Given any object a and any set A either $a \in A$ or $a \notin A$, but the two statements cannot hold simultaneously.
(ii) If $a = b$ then $a \in A$ implies $b \in A$ and $a \notin A$ implies $b \notin A$.
(iii) Sets A and B are equal if and only if $a \in A$ implies $a \in B$ and $a \in B$ implies $a \in A$.

AXIOM 1.4. A set is not equal to any object that is not a set.

AXIOM 1.5. Given a class of objects, there exists a set that has the objects of the given class as elements and that has no other elements.

AXIOM 1.6. Given arbitrary objects a, b, there exists an object, called an *ordered pair* (notation: $\langle a, b\rangle$), we require that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = b$ and $c = d$ and that no ordered pair is equal to an object that is not an ordered pair.

DEFINITION 1.1. Given an ordered pair $\langle a, b\rangle$, the object a is called the *first element* of the ordered pair, and the object b is called the *second element* of the ordered pair.

DEFINITION 1.2. Any set f of ordered pairs which is such that $\langle a, b\rangle \in f$, $\langle c, d\rangle \in f$ and $a = c$ implies $b = d$ is called a *function.* The set of all first elements of the ordered pairs of the function is called the *domain of definition* of the function. The set of all second elements of the ordered pairs of the function is called the *domain of values* of the function. Objects in the domain of definition are called *independent variables* and objects in the domain of values are called *dependent variables.*

DEFINITION 1.3. If $\langle a, b\rangle \in f$ then $f(a)$ is understood to be b.

THEOREM 1.1. *If a is in the domain of definition of a function f and if $a = c$, then $f(a) = f(c)$.*

Proof. $f(a) = b$ means $\langle a, b\rangle \in f$. Let $f(c) = d$ then $\langle c, d\rangle \in f$. Since $a = c$, it follows from the definition of a function that $\langle a, b\rangle = \langle c, d\rangle$, and hence $b = d$ and $f(a) = f(c)$.

DEFINITION 1.4. Let f be a function with A as domain of definition and B as domain of values. If f is such that $\langle a, b\rangle \in f$, $\langle c, d\rangle \in f$ and $b = d$ implies $a = c$, then f is said to define a *one-to-one correspondence* from A to B.

THEOREM 1.2. *If f defines a one-to-one correspondence from A to B, there exists a function g which defines a one-to-one correspondence from B to A.*

Outline of proof. g consists of all ordered pairs $\langle a, b\rangle$ such that $\langle b, a\rangle \in f$.

DEFINITION 1.5. Instead of writing one-to-one correspondence we write 1–1 correspondence; we also use the notation $A \leftrightarrow B$ to indicate that a 1–1 correspondence f exists between the sets A and B. The notation $a \leftrightarrow b$ is understood to mean that $\langle a, b\rangle$ is an element of this 1–1 correspondence f.

DEFINITION 1.6. The symbol $[a, b, \cdots]$ is used for a set which contains the elements a and b (i.e., $a \in A$, $b \in A$) as well as others. The symbol $[a, b]$ denotes the set which consists of the elements a and b only. Analogous interpretation is to be given to the symbols: $[a]$, $[a, \cdots]$, $[a, b, c]$, $[a, b, c, \cdots]$, etc.

REMARK 1.2. It should be noted that $[a]$ (the set consisting of the element a) and a are not equal.

REMARK 1.3. Whenever possible, we shall use capital letters for sets and lower-case letters for objects that are not sets or are not to be considered as such in the context except that functions will usually be denoted by lower-case letters.

DEFINITION 1.7. The set which is such that for every object a, $a \notin S$ is called the *null set.*

DEFINITION 1.8. A set A is said to be *contained in* a set B (notation: $A \subset B$) if, for every $a \in A$, $a \in B$. If $A \subset B$, A is called a *subset* of B, and, if in addition there exists a $b \in B$ such that $b \notin A$, then A is called a *proper subset* of B.

THEOREM 1.3. *$A = B$ if and only if $A \subset B$ and $B \subset A$.*

Proof. An immediate consequence of Ax. 1.3 and Def. 1.8.

THEOREM 1.4. If $A \subset B$ and $B \subset C$, then $A \subset C$.

Proof. Def. 1.8.

DEFINITION 1.9. If a 1–1 correspondence can be established between the elements of a set A and those of a set B, then A and B are said to have the *same cardinal number.*

DEFINITION 1.10. All sets that have the same cardinal number as the set $[a]$ are said to have the *cardinal number one* (or one element); all sets having the same cardinal number as the set $[a, b]$ are said to have the *cardinal number two* (or two elements); etc.

DEFINITION 1.11. Let S be a given set. A function having the set of all ordered pairs $\langle a, b\rangle$, $a \in S$, $b \in S$, as domain of definition and the set $[0, 1]$ as domain of values is said to define an *equivalence relation* on S provided: $f(\langle a, a\rangle) = 1$, $f(\langle a, b\rangle) = f(\langle b, a\rangle)$, if $f(\langle a, b\rangle) = 1$ and $f(\langle b, c\rangle) = 1$ then $f(\langle a, c\rangle) = 1$. If $f(\langle a, b\rangle) = 1$, then a is said to be *equivalent* to b (notation: $a \sim b$), whereas if $f(\langle a, b\rangle) = 0$, then a is said to be *not equivalent* to b (notation: $a \nsim b$).

THEOREM 1.5. *An equivalence relation satisfies the following properties:*

(i) $a \sim a$.

(ii) *if* $a \sim b$, *then* $b \sim a$.

(iii) *if* $a \sim b$ *and* $b \sim c$, *then* $a \sim c$.

Proof. Properties i, ii, and iii are immediate consequences of Def. 1.11.

DEFINITION 1.12. The properties i, ii, iii of an equivalence relationship are called, respectively, the *reflexive*, *symmetric*, and *transitive* property.

THEOREM 1.6. *In every set "equality" is an equivalence relationship.*

Proof. Left to the reader.

THEOREM 1.7. *If in a set S an equivalence relation is defined, this relation can be used to decompose S into mutually exclusive subsets (equivalence classes) which are such that elements of the same equivalence class are equivalent to each other while elements of distinct equivalence classes are not equivalent to each other.*

Proof. An equivalence class M_a is defined by $x \in M_a$ if and only if $x \sim a$. The transitive property then insures that, if $x \in M_a$ and $y \in M_a$, then $x \sim y$. If $M_a \neq M_b$ and $x \in M_a\, y \in M_b$, then $x \nsim y$ since $x \sim y$, the alternative, would imply $a \sim b$ and hence $M_a = M_b$.

REMARK 1.4. Th. 1.7 forms the basis for a not infrequently employed device of introducing new objects in mathematics (i.e., the equivalence classes). It essentially amounts to neglecting the differences between objects and emphasizing their common property and thus leads to the formation of general concepts. An example is provided by the concept of cardinal number which can be defined as the property common to all sets having the same cardinal number.

DEFINITION 1.13. A set S is called the *union* of sets A and B (notation: $S = A \cup B$) if $a \in S$ if $a \in A$ or $a \in B$.

DEFINITION 1.14. A set S is called the *intersection* of sets A and B (notation: $S = A \cap B$) if $a \in S$ if $a \in A$ and $a \in B$.

DEFINITION 1.15. Let S be a set. To every element α of S let there be assigned a set A_α. The set consisting of all those elements a which belong to A_α for at least one $\alpha \in S$ is called the union of the sets A_α and is denoted by

$$\bigcup_{\alpha \in S} A_\alpha.$$

DEFINITION 1.16. Let S be a set. To every element α of S let there be assigned a set A_α. The set consisting of those elements a which belong to A_α for every $\alpha \in S$ is called the intersection of the sets A_α and is denoted by

$$\bigcap_{\alpha \in S} A_\alpha.$$

THEOREM 1.8. $A \cup B = B \cup A.$

THEOREM 1.9. $A \cap B = B \cap A.$

THEOREM 1.10. $A \cup (B \cup C) = (A \cup B) \cup C.$

THEOREM 1.11. $A \cap (B \cap C) = (A \cap B) \cap C.$

THEOREM 1.12. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$

THEOREM 1.13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Proofs. Left to the reader.

DEFINITION 1.17. Let the set $A \subset S$; then B is called the *complement of A with respect to S* (reference to the embedding set is usually omitted, since it is usually clear from the context) and is denoted by A^* if $a \in B$ if $a \in S$, and $a \notin A$.

THEOREM 1.14. $(A^*)^* = A.$

THEOREM 1.15. $\bigcup_{\alpha \in S} A_\alpha^* = (\bigcap_{\alpha \in S} A_\alpha)^*.$

THEOREM 1.16. $\bigcap_{\alpha \in S} A_\alpha^* = (\bigcup_{\alpha \in S} A_\alpha)^*.$

DEFINITION 1.18. A function having as its domain of definition a subset of the set of all ordered pairs $\langle a, b\rangle$, $a \in S$, $b \in S$ and as its domain of values the set S is said to define a *binary operation* on S. Instead of writing $\langle\langle a, b\rangle, c\rangle \in f$, we write $a \circ b = c$.

DEFINITION 1.19. A set S is called *closed* under a binary operation defined on it if $a \circ b$ is defined for every $a \in S$, $b \in S$ and is defined to be an element of S.

DEFINITION 1.20. A binary operation $\circ$ is called *commutative* in S if

$$a \circ b = b \circ a$$

for all $a \in S$, $b \in S$ for which $a \circ b$ is defined.

DEFINITION 1.21. A binary operation is called *associative* in S if

$$a \circ (b \circ c) = (a \circ b) \circ c$$

for all $a \in S$, $b \in S$, $c \in S$ for which the indicated expressions are defined.

DEFINITION 1.22. A binary operation $\circ$ is said to be left (right) distributive with respect to another binary operation $\circ'$ in S if

$$a \circ (b \circ' c) = (a \circ b) \circ' (a \circ c),$$
$$((b \circ' c) \circ a = (b \circ a) \circ' (c \circ a))$$

for all $a \in S$, $b \in S$, $c \in S$ for which the above expressions are defined.

REMARK 1.5. Parentheses are customarily omitted in certain combinations of binary operations. Those combinations that are to be used in this book are: $-ab$, which means $-(ab)$, $ab \pm cd$, which means $(ab) \pm (cd)$, $\frac{a}{b} \pm \frac{c}{d}$, which means, $(a/b) \pm (c/d)$, $\frac{a \pm b}{c}$, which means $(a \pm b)/c$, and $\frac{a}{b \pm c}$, which means $a/(b \pm c)$.

THEOREM 1.17. *Let $\mathcal{S}$ be the set of all subsets of a given set S. In $\mathcal{S}$ the processes of forming unions and intersections are binary operations. These operations are commutative and associative, and each is distributive with respect to the other. Finally the set $\mathcal{S}$ is closed under both operations.*

Proof. The theorem follows from the appropriate definitions and Th. 1.8 to Th. 1.13.

Definition 1.23. Let certain binary operations be defined in the sets S and S'. If it is possible to establish a 1–1 correspondence between the sets S and S' and a 1–1 correspondence between the set of binary operations defined in S and the set of operations defined in S' and if these 1–1 correspondences satisfy the condition that, if $\circ \leftrightarrow \circ'$, $a \leftrightarrow a'$, $b \leftrightarrow b'$, then $a \circ b \leftrightarrow a' \circ' b'$, then the set S (together with the operations defined in it) is said to be *isomorphic* to the set S' (together with the operations defined in it).

Definition 1.24. If two sets are isomorphic (one usually speaks simply of isomorphic sets, it being assumed that it is clear from the context what are the operations in the two sets, with respect to which the sets are isomorphic), they are said to have the *same algebraic structure.*

Theorem 1.18. *Isomorphism is an equivalence relationship.*

Proof. Left to the reader.

Definition 1.25. Let $\mathcal{S}$ be the set of all subsets of a given set S. A function f having $\mathcal{S}$ as domain of definition and as domain of values is said to define a *topological structure* in S, provided $f(A) \supset A$ for every $A \in \mathcal{S}$. $f(A)$ is usually called the closure of A. The set S together with the function defining a topological structure in it is called a *general topological space.*

Definition 1.26. Let S and S' be general topological spaces. If a 1–1 correspondence can be established between S and S' in such a way that the 1–1 correspondence induced between $\mathcal{S}$ and $\mathcal{S}'$ is such that if $A \leftrightarrow A'$, $B \leftrightarrow B'$, where $B = f(A)$, then $B' = f'(A')$, then S and S' are said to be *homeomorphic* or to have the *same topological structure.* The functions f and f' are supposed to be the ones that define topological structures in S and S', respectively.

EXERCISES

1. Complete the proof of Th. 1.2.
2. Prove Th. 1.8.
3. Prove Th. 1.12.
4. Prove Th. 1.14.
5. Prove Th. 1.15.
6. Prove Th. 1.18.
7. Show how the terms "red," "chair" could be defined by equivalence classes.
8. What is the relation of the class "red chair" to the classes "red" and "chair"?
9. Show by examples, that the same set S can be made into a general topological space in many different ways.
10. Give examples of (a) noncommutative, (b) nonassociative binary operations.

REAL NUMBERS

DEFINITION 2.1. A *group* is a set G in which an associative binary operation $\circ$ is defined, under which G is closed, in which there exists an element e (the *identity* element) such that $a \circ e = e \circ a = a$ for all $a \in G$ and in which, for every $a \in G$, there exists an element $a^{-1} \in G$ (the *inverse* element) such that $a \circ a^{-1} = a^{-1} \circ a = e$.

DEFINITION 2.2. A *ring* is a set R in which a commutative binary operation is defined such that R is a group with respect to $\circ$, and in which moreover another associative binary operation $\circ'$ is defined which is distributive with respect to $\circ$ and under which R is closed.

DEFINITION 2.3. A *field* is a set F in which two commutative binary operations $\circ$ and $\circ'$ are defined. F is a group with respect to $\circ$. Moreover $\circ'$ is distributive with respect to $\circ$. Finally if e is the identity element of F with respect to $\circ$ then $F \cap [e]^*$ forms a group with respect to $\circ$.

DEFINITION 2.4. A field F is called an *ordered field* if an order relation is defined among its elements by the two symbols "$>$" and "$<$" which satisfies the following conditions:

(i) For any $a \in F$, $b \in F$, one and only one of the following statements holds: $a = b$, $a > b$, $a < b$.
(ii) If $a = c$, $b = d$ then $a > b$ implies $c > d$.
(iii) If $a > b$ and $c \geq d$ ($\geq$ means $>$ or $=$) then $a \circ c > b \circ d$.
(iv) If $a > b$ and $c > 0$ (the identity element of the operation $\circ$) then $a \circ' c > b \circ' c$.

DEFINITION 2.5. Let F be an ordered field. For certain subsets F_1 of F there exist elements $b_{f_1} \in F$ such that for all $x \in F_1$, $x \leq b_{f_1}$. The element b_{f_1} is then called *an upper bound* of F_1. If F has the property that every $F_1 \subset F$ which has an upper bound has a least upper bound $b^0_{f_1}$ (i.e., $b^0_{f_1}$ is an upper bound of F_1 and all upper bounds of F_1 satisfy $b_{f_1} \geq b^0_{f_1}$), then F is called a *complete ordered field*.

AXIOM 2.1. There exists a set, called the set of *real numbers*, which forms a complete ordered field.

Definition 2.6. In the set of real numbers the binary operation $\circ$ is written as "+" and called *addition*, its identity is denoted by 0 (*zero*) its inverse elements are written $-a$ (*minus* a). The operation $\circ'$ is written as "$\cdot$" and called *multiplication*, its identity is denoted by 1 (*one*) and its inverse elements are written as $1/a$. The symbols $>$ and $<$ are read as "*greater than*" and "*less than*," respectively. If $a > 0$, it is called a *positive* number, if $b < 0$, it is called a *negative* number.

Theorem 2.1. *If* $a = c$ *and* $b = d$, *then* $a + b = c + d$ *and* $a \cdot b = c \cdot d$.

Proof. $a + b = f(\langle a, b\rangle)$, $c + d = f(\langle c, d\rangle)$. Since $\langle a, b\rangle = \langle c, d\rangle$ (Ax. 1.6) it follows from Th. 1.1 that $a + b = c + d$. The proof that $ab = cd$ is analogous.

Theorem 2.2. *For any real numbers* a *and* b *the equation* $x + b = a$ *has a unique solution.*

Proof. $x = a + (-b)$ is a solution. This is seen as follows: $x + b = (a + (-b)) + b = a + ((-b) + b) = a + 0 = a$. Assume there existed another solution, say, y, then $y + b = a$, and hence $(y + b) + (-b) = a + (-b)$, but $(y + b) + (-b) = y + (b + (-b)) = y + 0 = y$, so that $y = a + (-b)$.

Theorem 2.3. *For any real numbers* a *and* $b \neq 0$ *the equation* $x \cdot b = a$ *has a unique solution.*

Remark 2.1. If no proof is given for a theorem, it means that the proof is left to the reader as an exercise.

Definition 2.7. If $x + b = a$, then x is called $a - b$. The operation thus defined is called *subtraction.*

Definition 2.8. If $xb = a$, $b \neq 0$, then x is called a/b. The operation thus defined is called *division.*

Theorem 2.4.
$$b + (-b) = (-b) + b = 0,$$
$$b \cdot (1/b) = (1/b) \cdot b = 1,\ b \neq 0.$$

Proof. Def. 2.6 and Def. 2.7 and Def. 2.8, respectively.

Theorem 2.5.
$$a + (-b) = a - b,$$
$$a \cdot (1/b) = a/b.$$

Proof. Th. 2.2 and Def. 2.7 for the first and Th. 2.3 and Def. 2.8 for the second relation.

Theorem 2.6. $-(-a) = a$.

Proof. Set $x = -(-a)$; then $x + (-a) = 0$. Hence $(x + (-a)) + a = 0 + a$. Now $0 + a = a$ and $(x + (-a)) + a = x + ((-a) + a) = x + 0 = x$. It follows that $x = a$.

Remark 2.2. Since the associative laws insure that $(a + b) + c = a + (b + c)$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, we can omit the parentheses and simply write $a + b + c$ and $a \cdot b \cdot c$, respectively. We did not do this earlier in order to show the importance of the associative laws. We use