Yoshio Kuramoto
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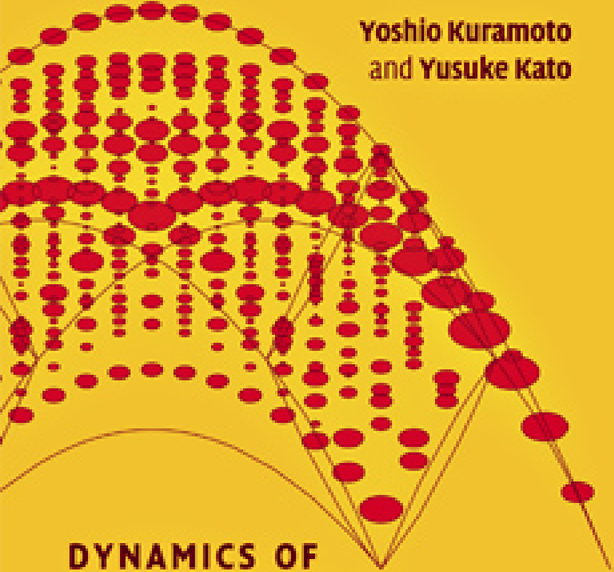

DYNAMICS OF
One-Dimensional Quantum Systems

Inverse-Square Interaction Models

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DYNAMICS OF ONE-DIMENSIONAL QUANTUM SYSTEMS

One-dimensional quantum systems show fascinating properties beyond the scope of the mean-field approximation. However, the complicated mathematics involved is a high barrier to non-specialists. Written for graduate students and researchers new to the field, this book is a self-contained account of how to derive a quasi-particle picture from the exact solution of models with inverse-square interparticle interactions.

The book provides readers with an intuitive understanding of exact dynamical properties in terms of exotic quasi-particles that are neither bosons nor fermions. Powerful concepts, such as the Yangian symmetry in the Sutherland model and its lattice versions, are explained. A self-contained account of non-symmetric and symmetric Jack polynomials is also given. Derivations of dynamics are made easier, and are more concise than in the original papers, so readers can learn the physics of one-dimensional quantum systems through the simplest model.

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Inverse-Square Interaction Models

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Preface

This book is concerned primarily with the exact dynamical properties of one-dimensional quantum systems. As a crucial property of exactly soluble models, we assume that the interaction decays as the inverse square of the distance. The family of these models is called the inverse-square interaction ($1/r^2$) models. In the one-dimensional continuum space, the model is often referred to as the Calogero–Sutherland model. In the one-dimensional lattice, on the other hand, the first $1/r^2$ models appeared as a spin model, which is now called the Haldane–Shastry model. Soon after the discovery of the Haldane–Shastry model, it was recognized that the imposition of supersymmetry allows the model to acquire the charge degrees of freedom, while keeping the exactly soluble nature. The resultant one-dimensional electron model is called the supersymmetric t–J model. Various generalizations of these models have been proposed.

Recent experimental progress in quasi-one-dimensional electron systems, especially by neutron scattering and photoemission spectroscopy, has enhanced the theoretical motivation for exploring the dynamics over a wide frequency and momentum range. The $1/r^2$ models are ideally suited to meet this situation, since the model allows derivation of exact dynamical information most easily and transparently. In spite of the special appearance of the $1/r^2$ models, the intuition thus obtained contributes greatly to understanding low-dimensional physics in general. This kind of approach to dynamics is complementary to another powerful approach using the bosonization and conformal field theory. The latter is especially suitable to asymptotics of correlation functions at long spatial and temporal distances.

The literature relevant to the $1/r^2$ models is vast and scattered. Moreover, many papers include a difficult-looking mathematical set-up. This situation may cause newcomers to see a barrier too high to jump over before enjoying the rich and beautiful ingredients of the $1/r^2$ models. For several years, the authors have realized the necessity of a comprehensive treatise. This book is intended to be accessible to non-specialists who are interested in strongly correlated quantum systems. It explains the wonderfully beautiful physics and related mathematics in a self-contained manner, without assuming special knowledge on theories in one dimension. In order to make a coherent discussion, we have included many results that are newly derived for this book, in addition to summarizing what has been reported in the literature. We hope that this book is useful not only to experts already working in the field, but also to graduate students and researchers trying to delve into the fascinating physics in low dimensions.

We are grateful to our collaborators, former students, and scientific colleagues who have worked in this area and helped our understanding of the subject, especially to M. Arikawa, N. Kawakami, T. Kimura, R. Nakai, O. Narayan, Y. Saiga, B. S. Shastry, B. Sutherland, T. Yamamoto, H. Yokoyama, and J. Zittartz. Our deepest thanks go to M. Arikawa, who carefully read the first version of the manuscript and made many useful suggestions.

1
Introduction

1.1 Motivation

Interactions in many-body systems bring about collective phenomena such as superconductivity and magnetism. In many cases, simple mean-field theory provides a basic understanding of these phenomena. In fermion systems in one dimension, however, neither the mean-field theory nor perturbation theory works if it starts from the non-interacting fermions. This is because the interaction effects in one dimension are much stronger than those in higher dimensions. Intuitively speaking, two particles cannot avoid collision in a single-way track in contrast with two and three dimensions. Thus the interaction effects appear in a drastic way in one dimension.

Another aspect in one dimension, which overcompensates the difficulty of perturbation and mean-field theories, is that a complete account of interaction effects is possible under certain conditions. The class of systems satisfying such conditions is referred to as exactly solvable. Soon after the establishment of quantum mechanics, Bethe solved exactly the Heisenberg spin model in one dimension [28]. The basic idea of the solution is now called the Bethe ansatz. Since then, theoretical physics in one dimension has developed into a magnificent edifice, including sophisticated mathematical techniques. In many cases, the eigenfunctions derived by the Bethe ansatz consist of plane waves that are defined stepwise for each spatial configuration of particles. Since the coefficients of plane waves depend on the configuration, the property of the wave function cannot be made explicit without detailed knowledge of these coefficients. We mention some of the recent monographs on the Bethe ansatz and its extensions [54, 118, 179]. A comprehensive account on exactly solvable models has recently been given by Sutherland [178].

The models solved by the Bethe ansatz are characterized by short-range interactions such as on-site repulsion or the next-nearest-neighbor exchange interaction. On the other hand, it was found by Calogero that another class of models also permits exact solution [34, 35]. The models have repulsive interactions decaying as the inverse square of the interparticle distance r. In order to prevent the blow-up of particles toward infinite distance, an attractive harmonic potential can be added to the system. Alternatively, one takes the periodic boundary condition with the system length L, and employs superposition of the $1/r^2$ potential as

$$\sum_{n=-\infty}^{\infty} \frac{1}{(r+nL)^2} = \left(\frac{\pi/L}{\sin \pi r/L}\right)^2. \tag{1.1}$$

Then by construction the system does not blow up, while keeping the translational invariance. This model was proposed by Sutherland [172, 174], and hence is called the Sutherland model. If one refers to both models simultaneously, it seems appropriate to call them the Calogero–Sutherland models. Some years later, Moser analyzed the classic mechanical version of these models mathematically [135], and his name is sometimes added in referring to the models.

The $1/r^2$ models have much simpler mathematical (algebraic) structure, compared to the conventional integrable models solved by the Bethe ansatz. This simplicity enables us to derive explicitly the exact expressions of dynamical correlation functions such as the Green function, the density–density correlation function, and the spin–spin correlation function. The resultant expressions are remarkably simple, but still keep nontrivial features inherent to interacting particle systems. Further, the mathematical tools used in the derivation are far from complicated. Thus, the $1/r^2$ models provide comprehensible examples for studying dynamics of interacting particles.

In contrast with the Fermi liquid in three dimensions, the one-dimensional fermions behave as the Tomonaga–Luttinger liquid in the limit of long time and long distance. Here the conformal field theory (CFT) describes nicely the asymptotics of correlation functions. According to the CFT, characterization of the interaction parameters can be done through analysis of the finite-size correction of the ground state energy. Since the $1/r^2$ models allow for calculation of the finite-size correction much more easily than the Bethe-solvable models, the $1/r^2$ models serve as an instructive example to visualize how the CFT works in the Tomonaga–Luttinger liquid. The importance of the $1/r^2$ models does not, however, lie only in the mathematical structure. Through the study of the $1/r^2$ models, one can also learn about the dynamics of the correlated electrons in real systems. For example, the neutron scattering intensity of $S = 1/2$ antiferromagnetic spin chain reveals a similarity to the spectral function of the spin correlation function of the $1/r^2$ exchange interaction model, which is called the Haldane–Shastry model [77, 161]. A related model with charge degrees of freedom is still exactly solvable provided a supersymmetry is imposed [119]. The spin–charge separation of one-dimensional electrons can then be explicitly seen in the spectral weight of the Green function of the supersymmetric t–J model.

1.2 One-dimensional interaction as a disguise

As we shall explain in detail, the wave function in the ground state of the $1/r^2$ models can be derived explicitly as the product of two-body wave functions. This feature is quite in contrast with cases solved by the Bethe ansatz. The special feature of the $1/r^2$ interaction already appears in most elementary quantum mechanics. Let us consider a free particle with mass $m = 1/2$ in the three-dimensional space. The Hamiltonian is given by

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\boldsymbol{l}^2}{r^2}, \tag{1.2}$$

where $r^2 = x^2 + y^2 + z^2$, and $\boldsymbol{l}$ is the angular momentum operator. We take the units $\hbar = 1$ throughout the book. In the polar coordinates, there appears a fictitious potential leading to the centrifugal force. Namely, the free motion in higher dimensions generates a fictitious potential if the radial motion alone is extracted [146]. Conversely, the potential $l(l+1)/r^2$ in the radial coordinate is a disguise of free motion in higher dimensions. The form of the radial kinetic energy in (1.2) is interpreted as coming from the metric of the one-dimensional space. Pursuing this idea in many-body systems, one gains a perspective that interactions in exactly solvable models are a disguise of some kind of free motion in another space [146]. Alternatively, a matrix model has been constructed where the coordinates of N particles are regarded as eigenvalues of an $N \times N$ matrix. The transformation matrix for diagonalization appears as the $1/r^2$ potential [151].

In the early stage of the Tomonaga–Luttinger theory, all low-energy excitations are regarded as bosons. Actually, the statistics of excitations need not be restricted to bosons. In some cases, the interaction among bosons is absorbed into a new statistics describing exclusion of available one-body states. This idea applies to many interacting systems approximately, and to the $1/r^2$ models exactly. The exclusion includes fermions and bosons as special cases. Generally, however, the statistics is fractional. In order to account for the resultant quasi-particles obeying fractional exclusion statistics, concepts such as the Yangian symmetry and the supersymmetry turn out to be useful. These new concepts make it much easier to understand exact dynamics (and also thermodynamics) intuitively. Our key strategy in this respect is to rely on the picture of quasi-particles obeying fractional exclusion statistics. In terms of these exotic quasi-particles, the dynamics of one-dimensional systems can be understood intuitively.

In the last decades, intensive study of the $1/r^2$ models has brought about deep intuition into the structure of the excitation spectrum in one-dimensional systems in general. The most remarkable observation is that elementary excitations behave as free particles subject to certain statistical constraints. As a result, these particles obey the statistics of neither fermions nor bosons. In other words, the exchange of two excitations leads to a scattering phase shift which is independent of their momenta, but which is neither π (antisymmetric) nor 0 (symmetric).

The situation may become clearer if we make an analogy to the Fermi liquid theory. The excitations in the Bethe-soluble models have a phase shift that does depend on their momenta. Therefore, certain parameters are necessary to characterize the momentum dependence. These parameters are analogous to Landau parameters that describe interactions between the quasi-particles in the Fermi liquid. In this analogy, the excitations in the $1/r^2$ models do not need the analogue of the Landau parameters, and are comparable to free fermions except for the statistics. Just as the understanding of metals in general has been much facilitated by the free-electron model, the dynamics in one dimension should be much better understood by reference to "free" models, i.e., the $1/r^2$ models.

1.3 Two-body problem with $1/r^2$ interaction

We demonstrate the peculiar features of the $1/r^2$ model by taking the simplest example. Let us consider the two-body problem with Hamiltonian

$$H_2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + g\left[\frac{\pi/L}{\sin\pi(x_1 - x_2)/L}\right]^2. \tag{1.3}$$

For the moment we assume that the two particles are distinguishable, and do not care about the symmetry of the wave function. If the distance $|x_1 - x_2|$ is much smaller than L, the interaction reduces to $g/(x_1 - x_2)^2$. The center of gravity $X = (x_1 + x_2)/2$ has free motion with wave number Q. In terms of X and the relative coordinate $x = x_1 - x_2$, the wave function is factorized into the form $\psi_g(x_1, x_2) = \phi_g(x)\exp(\mathrm{i}QX)$, where $\phi_g(x)$ is an eigenfunction of a one-body Hamiltonian H_1 given by

$$H_1(x) = H_2 - \frac{1}{2}Q^2 = -2\frac{\partial^2}{\partial x^2} + g\left(\frac{\pi/L}{\sin \pi x/L}\right)^2. \tag{1.4}$$

Instead of solving (1.4) in the standard way, we discuss alternative ideas which are useful in generalizing to the many-body problem. Let us first examine the wave function $\phi_g(x)$ for $|x| \ll L$ where the potential in H_1 tends to g/x^2. Then $H_1(x)$ has the scaling property

$$H_1(ax) = a^{-2}H_1(x).$$

An eigenfunction should also have the scaling property for $x \sim 0$

$$\phi_g(ax) = a^\lambda \phi_g(x), \tag{1.5}$$

with certain number λ. The only solution with property (1.5) is the power-law function $\phi_g(x) = x^\lambda$. Upon differentiation twice, we obtain $\lambda(\lambda-1)\phi_g(x)/x^2$. By taking $\lambda(\lambda-1) = g/2$, the kinetic term cancels the potential term. Then $\phi_g(x)$ turns out to be the eigenfunction of H_1. Since we have $\lambda = (1 \pm \sqrt{1+2g})/2$, only the case of $g \geq -1/2$ is meaningful. Otherwise, the attractive potential causes the system to collapse as in the classical system, and the ground state cannot be defined. This situation has already been discussed by Landau and Lifshitz [122] and by Sutherland [172]. In the following we only consider the case $g > 0$, and take the positive λ as the relevant solution. We can extend the range of x so as to be consistent with the periodic boundary condition, simply by replacing x^α by $|\sin \pi x/L|^\alpha$.

It is possible to derive all the eigenvalues and eigenfunctions by using the factorization method [89], which has been refined under the name of "supersymmetric quantum mechanics" [192]. We introduce a variable $\eta \equiv \pi x/L$ and rewrite (1.4) as

$$H_1 = 2\left(\frac{L}{\pi}\right)^2 \left[p_\eta^2 + W_\lambda(\eta)^2 + W'_\lambda(\eta) + \lambda^2\right] \equiv 2\left(\frac{L}{\pi}\right)^2 \mathcal{H}_\lambda, \tag{1.6}$$

where $p_\eta = -\mathrm{i}\partial/\partial\eta$ and $W_\lambda(\eta) = \lambda \cot \eta$. Then $\mathcal{H}_\lambda$ takes a factorized form

$$\mathcal{H}_\lambda = (p_\eta - \mathrm{i}W_\lambda)(p_\eta + \mathrm{i}W_\lambda) + \lambda^2 \equiv A_\lambda^\dagger A_\lambda + \lambda^2. \tag{1.7}$$

An eigenfunction of $\mathcal{H}_\lambda$ is given by

$$\phi_\lambda(\eta) = \sin^\lambda \eta = \exp[U_\lambda(\eta)], \tag{1.8}$$

where we have introduced $U_\lambda(\eta) = \lambda \ln \sin \eta$. This gives $U'_\lambda(\eta) = W_\lambda(\eta)$, and it is evident that $A_\lambda\phi_\lambda(\eta) = 0$. Since $A_\lambda^\dagger A_\lambda$ is a non-negative operator, there are no states with lower energy. Hence, ϕ_λ gives the ground state of $\mathcal{H}_\lambda$ with energy λ^2.

We note the property

$$A_\lambda A_\lambda^\dagger = p_\eta^2 + W_\lambda(\eta)^2 - W'_\lambda(\eta) = p_\eta^2 + \frac{\lambda(\lambda+1)}{\sin^2\eta} - \lambda^2$$
$$= A_{\lambda+1}^\dagger A_{\lambda+1} - \lambda^2, \tag{1.9}$$

which corresponds to the shift $\lambda \to \lambda + 1$ in $\mathcal{H}_\lambda$ together with subtracting the constant term λ^2. Combination of (1.7) and (1.9) makes it possible to derive all the excited states. Let us take the ground state $\phi_{\lambda+1}$ of $H_{\lambda+1}$ with the eigenvalue $(\lambda+1)^2$. Namely, we have

$$A_\lambda A_\lambda^\dagger \phi_{\lambda+1} = [(\lambda+1)^2 - \lambda^2]\phi_{\lambda+1}. \tag{1.10}$$

Applying $A_\lambda^\dagger$ from the left, we obtain

$$A_\lambda^\dagger A_\lambda A_\lambda^\dagger \phi_{\lambda+1} = [(\lambda+1)^2 - \lambda^2]A_\lambda^\dagger \phi_{\lambda+1}. \tag{1.11}$$

Thus the state $A_\lambda^\dagger \phi_{\lambda+1}$ proves to be an excited state of $A_\lambda^\dagger A_\lambda$.

We now explain briefly the idea of the supersymmetric quantum mechanics. We may treat the pair $A_\lambda A_\lambda^\dagger$ and $A_\lambda^\dagger A_\lambda$ as components of a 2×2 matrix:

$$\mathcal{H}_{\text{pair}} = \begin{pmatrix} A_\lambda^\dagger A_\lambda & 0 \\ 0 & A_\lambda A_\lambda^\dagger \end{pmatrix} = QQ^\dagger + Q^\dagger Q \equiv \{Q, Q^\dagger\}, \tag{1.12}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ A_\lambda & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A_\lambda^\dagger \\ 0 & 0 \end{pmatrix}. \tag{1.13}$$

The space of the 2×2 matrix can be regarded as a pseudo-spin spanned by the Pauli matrices. Here Q and $Q^\dagger$ have an analogy with spin-flips $s_\pm = s_x \pm \mathrm{i}s_y$. Alternatively, we may include the pseudo-fermion operators $f, f^\dagger$ by the identification

$$\frac{1}{2}(1 - \sigma_z) = f^\dagger f. \tag{1.14}$$

Then the operators $Q, Q^\dagger$ in (1.12) are written as

$$Q = f^\dagger A_\lambda, \quad Q^\dagger = A_\lambda^\dagger f. \tag{1.15}$$

It is obvious that $Q^2 = (Q^\dagger)^2 = 0$ and

$$[\mathcal{H}_{\text{pair}}, Q] = [\mathcal{H}_{\text{pair}}, Q^\dagger] = 0. \tag{1.16}$$