Ergebnisse der Mathematik und ihrer Grenzgebiete
3. Folge · Band 22
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Gerd Faltings Ching-Li Chai

Degeneration of Abelian Varieties

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Foreword

The topic of this book is the theory of degenerations of abelian varieties and its application to the construction of compactifications of moduli spaces of abelian varieties. These compactifications have applications to diophantine problems and, of course, are also interesting in their own right.

Degenerations of abelian varieties are given by maps $G \to S$ with S an irreducible scheme and G a group variety whose generic fibre is an abelian variety. One would like to classify such objects, which, however, is a hopeless task in this generality. But for more specialized families we can obtain more: The most important theorem about degenerations is the stable reduction theorem, which gives some evidence that for questions of compactification it suffices to study semi-abelian families; that is, we may assume that G is smooth and flat over S, with fibres which are connected extensions of abelian varieties by tori. A further assumption will be that the base S is normal, which makes such semi-abelian families extremely well behaved. In these circumstances, we give a rather complete classification in case S is the spectrum of a complete local ring, and for general S we can still say a good deal.

For a complete base $S = \mathrm{Spec}(R)$ (R a complete and normal local domain) the main result about degenerations says roughly that G is (in some sense) a quotient of a covering $\tilde{G}$ by a group of periods. Here $\tilde{G}$ is globally an extension of an abelian scheme A over R by a torus T/R, and the periods form a free abelian group in $\tilde{G}(K)$, K the fraction field of R. To make this more precise we should explain how to form such quotients:

The first example is the Tate-curve, where $\tilde{G} = G_m$ is the multiplicative group, and the periods are of the form $q^{\mathbf{Z}}$, $q \in R$ a non-unit different from zero. To form the quotient $G = \tilde{G}/q^{\mathbf{Z}}$ embed $\tilde{G}$ equivariantly into the projective line $\mathbb{P}^1_R$ over R, with coordinate T. Then blow up the closed subschemes defined by $\{q = 0, T = 0\}$ and $\{q = 0, T = \infty\}$. The resulting R-scheme is still equal to the old projective line over $R[1/q]$, while over $R/(q)$ we have obtained a chain of three copies of $\mathbb{P}^1$ glued together by identifying 0 in one copy to ∞ in the next. Blowing up the "end points" and going on we obtain an R-scheme $\tilde{P}$ (not of finite type) which over $R[1/q]$ (respectively over $R/(q)$) is equal to $\mathbb{P}^1$ (respectively an infinite chain of $\mathbb{P}^1$'s). $\tilde{G}$ still operates on $\tilde{P}$, and multiplication by q (over $R[1/q]$, where q defines a section of $\tilde{G}$) extends to an automorphism of $\tilde{P}$ which over $\{q = 0\}$ shifts the infinite chain of $\mathbb{P}^1$'s by one. It follows that we may form the quotient under q-multiplication of the formal scheme $\tilde{P}_{\text{for}}$ associated to $\tilde{P}$, which defines a formal scheme P_{for} over R. Over $\{q = 0\}$ it