Geometriae Dedicata **58**: 53–62, 1995.

4-Dimensional Compact Projective Planes with Small Nilradical

Dedicated to Prof. H. Salzmann on the occasion of his 65th birthday

HAUKE KLEIN
Mathematisches Seminar, Universität Kiel, Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany

(Received: 7 October 1994)

Abstract. We consider 4-dimensional compact projective planes with a solvable 6-dimensional collineation group Σ and with orbit type $\geq (2,1)$, i.e. Σ fixes a flag $v \in W$, acts transitively on $\mathcal{L}_v \backslash \{W\}$ and fixes no point in the set $W \backslash \{v\}$. We prove a series of lemmas concerning the action of invariant subgroups of Σ. These lemmas are applied to prove that the maximal connected nilpotent invariant subgroup of Σ has dimension at least 4.

Mathematics Subject Classifications (1991): 51H10, 51H20.

1. Introduction

In the fundamental papers ([16], [17]), Salzmann studies 4-dimensional compact projective planes, i.e. topological projective planes with point space homeomorphic to the classical plane over the complex numbers. We denote by Σ the connected component of the group of all continuous collineations of a 4-dimensional compact projective plane π. The plane is called flexible if Σ has an open orbit in the space of flags. Since this space has dimension 6 we have $\dim \Sigma \geq 6$ for such a plane. The group Σ is known to be a Lie group with $\dim \Sigma \leq 16$. All planes with $\dim \Sigma \geq 7$ are completely classified ([4], [13]). All flexible translation planes (and hence also the dual translation planes) are classified ([2]) and also all flexible shift planes ([1], [11], [12], [3]). By [11] the plane π is a shift plane, a translation plane or a dual translation plane if and only if the group Σ contains a subgroup isomorphic to the vector group $\mathbf{R}^4$. As a general reference for 4-dimensional compact projective planes see Chapter 7 of [18].

In this paper we will consider the remaining case that there is no 4-dimensional abelian subalgebra in $\ell(\Sigma)$ and $\dim \Sigma = 6$. If Σ is not solvable then $\Sigma \cong \mathrm{GL}_2^+\mathbf{R} \ltimes \mathbf{R}^2$ ([13]). Here we assume that Σ is solvable. By [5], the group Σ fixes a flag $v \in W$. The cases that Σ fixes either a further line $Y \in \mathcal{L}_v \backslash \{W\}$ or a further point $u \in W \backslash \{v\}$ or that Σ is transitive neither on $\mathcal{L}_v \backslash \{W\}$ nor on $W \backslash \{v\}$ are already done ([5], [8], [9]). Hence we assume in the following:

Σ has an orbit type $\geq (2,1)$ in the sense of [5], i.e. Σ does not fix a line $Y \in \mathcal{L}_v \backslash \{W\}$ and dually Σ fixes no point $u \in W \backslash \{v\}$ and acts transitively on $\mathcal{L}_v \backslash \{W\}$ or on $W \backslash \{v\}$. Up to duality we may assume that Σ acts transitively on $\mathcal{L}_v \backslash \{W\}$. Next we denote by N the nilradical of Σ, i.e. the maximal connected nilpotent invariant subgroup of Σ. By [14] we know $\dim N \geq 3$. All planes with $\dim N \geq 5$ are known ([6], [7]).

By [14], a 6-dimensional solvable Lie algebra with a 3-dimensional nilradical is isomorphic to either l_2^3 or to $l_2^{\mathbf{C}} \times l_2$, where l_2 denotes the 2-dimensional, non-abelian Lie algebra and $l_2^{\mathbf{C}}$ is the analogous complex Lie algebra considered as a 4-dimensional real algebra.

In order to exclude the possibility $\ell(\Sigma) \cong l_2^3$ we consider the base ideal of $\ell(\Sigma)$, i.e. the ideal generated by all 1-dimensional ideals of $\ell(\Sigma)$ ([10, II.5]). We denote the corresponding invariant subgroup of Σ by $B(\Sigma)$. The Lie algebra l_2^3 has a big base ideal and in Section 2 we will see that this leads to a contradiction.

The most frequently used lemma on the structure of the set of fixed points of collineations $\phi \in \Sigma$ is the lemma on quadrangles: *If $\phi \in \Sigma$ fixes a quadrangle, i.e. four points, no three of which are collinear, then $\phi = 1$.*

The organization of this paper is as follows. In Section 2 we prove a series of lemmas under the assumption of orbit type $\geq (2,1)$. These results hold independently of the structure of the nilradical N and they seem to be useful for the general classification in orbit type $\geq (2,1)$. In Section 3 we describe the subalgebra structure of the Lie algebra $l_2^{\mathbf{C}} \times l_2$. This algebra occurs in Section 4 as the Lie algebra of Σ in the hypothetic case $\dim N = 3$. In Section 4 we will exclude the group of type $l_2^{\mathbf{C}} \times l_2$, and thus prove that the nilradical N of Σ has dimension at least 4.

2. General Facts in Orbit Type $\geq (2,1)$

Let $\pi = (\mathcal{P}, \mathcal{L})$ be a 4-dimensional compact projective plane and denote by Σ the connected component of the group of all continuous collineations of π. We assume the following conditions:

- π is neither a translation plane nor a dual translation plane, nor a shift plane, i.e. Σ contains no subgroup isomorphic to the vector group $\mathbf{R}^4$.
- The group Σ is 6-dimensional and solvable. By [5], Σ fixes a flag $v \in W$.
- Σ acts transitively on $\mathcal{L}_v \backslash \{W\}$ and fixes no point of the set $W \backslash \{v\}$.

In particular, Σ fixes no point in $\mathcal{P} \backslash \{v\}$ and no line in $\mathcal{L} \backslash \{W\}$. We prove a series of lemmas in this situation.

LEMMA 1. *Let $1 \neq N \trianglelefteq \Sigma$. Then N fixes no point in $\mathcal{P} \backslash W$ and no line in $\mathcal{L} \backslash \mathcal{L}_v$. If a line $l \in \mathcal{L} \backslash \{W\}$ contains an orbit of N on $\mathcal{P} \backslash W$ then $v \in l$, and dually if a pencil $\mathcal{L}_u (u \in \mathcal{P} \backslash \{v\})$ contains an orbit of N on $\mathcal{L} \backslash \mathcal{L}_v$ then $u \in W$.*

Proof. Assume N fixes an affine point $o \in \mathcal{P}\backslash W$. Since the group Σ does not fix the line $o \vee v$, there exists another line in $\mathcal{L}_v$ containing a fixed point of N, i.e. there is a point $o' \in \mathcal{P}\backslash W$ which is fixed under N and which is not contained in $o \vee v$. Since N fixes the point $(o \vee o') \wedge W \in W\backslash\{v\}$ there exists a further fixed point $u \in W\backslash\{v\}$. Hence N fixes a quadrangle, so $N = 1$. The statement about line orbits is proved dually. Assume the line $l \in \mathcal{L}\backslash\{W\}$ contains an orbit $\mathcal{B}$ of N in $\mathcal{P}\backslash W$. Since $|\mathcal{B}| \geq 2$, the group N necessarily fixes l, and we get $v \in l$. The last statement is proved dually.

LEMMA 2. *Σ acts effectively on every orbit on $\mathcal{P}\backslash W$ and on $\mathcal{L}\backslash\mathcal{L}_v$.*

Proof. Let $\mathcal{B} \subseteq \mathcal{L}\backslash\mathcal{L}_v$ be an orbit of Σ. Denote by N the kernel of the action of Σ on $\mathcal{B}$ and apply Lemma 1.

LEMMA 3. *Let $N \trianglelefteq \Sigma$ be a connected invariant subgroup of Σ with* $\dim N \leq 2$. *Assume $N \subseteq \Sigma_{[v,v]}$. Then $N \subseteq \Sigma_{[v,W]}$ and if* $\dim N = 2$ *then $N = \Sigma_{[v,W]}$.*

Proof. In the case $\dim N = 1$ the elements of N have a common axis and a common center and these are fixed under Σ, hence they are equal to W and v respectively. Assume $\dim N = 2$. The set $\mathcal{A} := \{Y \in \mathcal{L}_v : N_{[Y,v]} \neq 1\}$ is invariant under the action of Σ. We have to show that $\mathcal{A} = \{W\}$. Assume the contrary, i.e. $\mathcal{A} \cap (\mathcal{L}_v\backslash\{W\}) \neq \emptyset$. Since Σ acts transitively on $\mathcal{L}_v\backslash\{W\}$ this implies $\mathcal{L}_v\backslash\{W\} \subseteq \mathcal{A}$. The space P of 1-parameter subgroups of N is a circle and the mapping ϕ which assigns to a 1-parameter subgroup $H \in P$ the common axis of its elements is continuous. Since N is the union of its 1-parameter subgroups we have $\mathcal{A} = \phi(P)$. Since $\mathcal{A}$ contains a subspace homeomorphic to the plane $\mathbf{R}^2$, the mapping ϕ cannot be one-to-one, i.e. there are $H_1, H_2 \in P$ with $H_1 \neq H_2$ and $\phi(H_1) = \phi(H_2)$. Hence the group $N = \langle H_1, H_2 \rangle$ has a common axis. But this contradicts $\mathcal{L}_v\backslash\{W\} \subseteq \mathcal{A}$. The last assertion is clear.

LEMMA 4. *Let $N \trianglelefteq \Sigma$ be an invariant subgroup with non-trivial centralizer $C_\Sigma N \neq 1$. Suppose that N consists of perspectivities.*

1. *Let z and A be the center and the axis of an element $\alpha \in N\backslash\{1\}$ respectively. Then $v \in A$ and $z \in W$.*
2. *$N \subseteq \Sigma_{[W,W]}$ or $C_\Sigma N \subseteq \Sigma_{[v,v]}$.*

Proof. (1) Apply Lemma 1 to the invariant subgroup $C_\Sigma N$.

(2) Assume $N \neq 1$. Let $\mathcal{A}$ be the set of all axes of elements $\neq 1$ of N. By (1), $\mathcal{A} \subseteq \mathcal{L}_v$ and $\mathcal{A}$ is invariant under the action of Σ, hence $\mathcal{A} = \{W\}$ or $\mathcal{L}_v\backslash\{W\} \subseteq \mathcal{A}$. In the first case we have $N \subseteq \Sigma_{[W]}$ and by an application of (1), $N \subseteq \Sigma_{[W,W]}$. Now assume $\mathcal{L}_v\backslash\{W\} \subseteq \mathcal{A}$. The centralizer $C_\Sigma N$ fixes the elements of the set $\mathcal{A}$, i.e. $C_\Sigma N \subseteq \Sigma_{[v]}$. A final application of part (1) to $C_\Sigma N$ yields $C_\Sigma N \subseteq \Sigma_{[v,v]}$.

LEMMA 5. *Let $N \trianglelefteq \Sigma$ be an invariant subgroup and assume $C_{\Sigma_o} N \neq 1$ for some affine point $o \in \mathcal{P}\backslash W$. Then $N \subseteq \Sigma_{[v,v]}$.*

Proof. Assume $N \neq 1$. The centralizer $C_{\Sigma_o}N$ fixes each point of the orbit o^N. Assume o^N is not contained in a line. Then there exists a point $o' \in o^N \backslash (o \vee v)$. Set $b := (o \vee o') \wedge W \in W \backslash \{v\}, X := o \vee o' = o \vee b$ and $Y := o \vee v$. Since $o^N \not\subseteq X$ there exists a point $o'' \in o^N \backslash X$. Since $C_{\Sigma_o}N$ fixes the points o, v, b, o'' the lemma on quadrangles implies $o'' \in Y$. But then $C_{\Sigma_o}N$ fixes the quadrangle $o, v, b, (o'' \vee b) \wedge (o' \vee v)$; a contradiction. This shows that o^N is contained in a line and Lemma 1 yields $o^N \subseteq o \vee v =: Y$ and $Y^N = Y$. Since Σ acts transitively on $\mathcal{L}_v \backslash \{W\}$ this implies $N \subseteq \Sigma_{[v]}$. By Lemma 4, $N \subseteq \Sigma_{[v,v]}$.

LEMMA 6. *The center of* Σ *consists entirely of perspectivities with center* v *and axis* W, *i.e.* $Z(\Sigma) \subseteq \Sigma_{[v,W]}$.

Proof. Since $\Sigma_o \neq 1$ for each point $o \in \mathcal{P} \backslash W$, Lemma 5 implies $Z(\Sigma) \subseteq \Sigma_{[v,v]}$. Since the axis of an arbitrary element $\zeta \in Z(\Sigma) \backslash \{1\}$ is fixed under Σ, the axis of such an ζ is W, i.e. $Z(\Sigma) \in \Sigma_{[v,W]}$.

LEMMA 7. *Let* $N \trianglelefteq \Sigma$ *be a* 1*-dimensional connected invariant subgroup. Then* $N \subseteq \Sigma_{[v,v]}$.

Proof. Choose an affine point $o \in \mathcal{P} \backslash W$. Since $\dim \Sigma_o \geq 2$ and $\dim C_\Sigma N \geq 5$, we have $\Sigma_o \cap C_\Sigma N \neq 1$. Lemma 5 implies $N \subseteq \Sigma_{[v,v]}$.

LEMMA 8. *We have* $B(\Sigma) \subseteq \Sigma_{[v,W]}$ *and, in particular,* $\dim B(\Sigma) \leq 2$.

Proof. By Lemma 7 each connected 1-dimensional invariant subgroup is contained in $\Sigma_{[v,v]}$ and Lemma 3 implies that these groups are even contained in $\Sigma_{[v,W]}$.

3. The Lie Algebra $l_2^{\mathbb{C}} \times l_2$

We consider the 6-dimensional solvable real Lie algebra $\mathcal{G} = l_2^{\mathbb{C}} \times l_2$. The right-hand factor l_2 is given as $l_2 = \langle e, f \rangle$ with $[e, f] = e$. The left-hand factor $l_2^{\mathbb{C}}$ may be generated by the matrices

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}.$$

These matrices satisfy

$$[R, S] = [B, C] = 0, [B, R] = B, [B, S] = C, [C, R] = C \text{ and } [C, S] = -B.$$

The Lie algebra $\mathcal{G}$ has exactly one 1-dimensional ideal, namely $\langle e \rangle$ and exactly two 2-dimensional ideals, namely $\langle e, f \rangle$ and $\langle B, C \rangle$. In particular, the full automorphism group Γ of $\mathcal{G}$ must fix the real l_2, and consequently it must fix the centralizer of this subalgebra too, i.e. $l_2^{\mathbb{C}}$ is invariant under Γ. This implies $\Gamma = \mathrm{Aut}(l_2^{\mathbb{C}}) \times \mathrm{Aut}(l_2)$.

The automorphisms of l_2 are $e \mapsto \alpha e, f \mapsto \beta e + f$ with $\alpha, \beta \in \mathbf{R}, \alpha \neq 0$, and such an automorphism is inner if and only if $\alpha > 0$.

We describe the automorphisms of the (real) Lie algebra $l_2^{\mathbf{C}}$. Write (z, w) for the element $aR + bS + cB + dC$ of $l_2^{\mathbf{C}}$ with $z = a + ib$ and $w = c + id$. The inner automorphisms are the mappings $(z, w) \mapsto (z, za + wb)$ with $a, b \in \mathbf{C}, b \neq 0$ and the outer automorphisms are $(z, w) \mapsto (\bar{z}, \bar{z}a + \bar{w}b)$ with $a, b \in \mathbf{C}, b \neq 0$. By a routine calculation it is possible to determine all subalgebras of $\mathcal{G}$ up to the action of Γ. The resulting list is given in Table I.

The ideals of $\mathcal{G}$ not containing $\mathcal{G}'$ are exactly: $\langle e \rangle = l_2', l_2, \langle B, C \rangle = (l_2^{\mathbf{C}})', l_2^{\mathbf{C}}$ and those subalgebras of $l_2^{\mathbf{C}}$ which contain $(l_2^{\mathbf{C}})'$. The simply connected Lie group with Lie algebra $l_2^{\mathbf{C}}$ is the group

$$\widetilde{L_2^{\mathbf{C}}} := \{(z, b) \colon z, b \in \mathbf{C}\},$$

with multiplication

$$(z_1, b_1) \cdot (z_2, b_2) = (z_1 + z_2, e^{z_2} b_1 + b_2).$$

The 1-parameter subgroups of this group corresponding to R, S, B, C are given by:

$$R(t) = (t, 0), S(t) = (it, 0), B(t) = (0, t) \quad \text{and} \quad C(t) = (0, it),$$

respectively. The center is

$$Z = \{(2\pi i n, 0) \colon n \in \mathbf{Z}\},$$

and consequently the Lie groups with Lie algebra $l_2^{\mathbf{C}}$ are enumerated by a natural number $n \in \mathbf{N} \cup \{\infty\}$. We write $L_{2,\infty}^{\mathbf{C}} = \widetilde{L_2^{\mathbf{C}}}$ and for each $n \in \mathbf{N}$:

$$L_{2,n}^{\mathbf{C}} = \{(z, b) \colon z \in \mathbf{C} \backslash \{0\}, b \in \mathbf{C}\}$$

with multiplication

$$(z_1, b_1) \cdot (z_2, b_2) = (z_1 z_2, z_2^n b_1 + b_2).$$

In particular $L_{2,1}^{\mathbf{C}}$ is the ordinary complex $L_2^{\mathbf{C}}$.

4. Excluding a 3-Dimensional Nilradical

In this section we will prove the following theorem:

THEOREM. *Let π be a 4-dimensional compact projective plane and denote by Σ the connected component of the group of continuous collineations of π. Assume*

TABLE I.

2-dimensional subalgebras

abelian	*non-abelian*
$\langle aR+bS, f\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$	$\langle e, aR+bS+f\rangle \ (a,b\in\mathbf{R})$
$\langle B, f\rangle$	$\langle e, B+f\rangle$
$\langle aR+bS, e\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$	$\langle B, R+yf\rangle \ (y\in\mathbf{R})$
$\langle bS+f, R+b'S\rangle \quad (b,b'\in\mathbf{R}, b\neq 0)$	$\langle B, R+e\rangle$
$\langle aR+bS, R+e\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$	$\langle B+e, R+f\rangle$
$\langle S, R+f\rangle$	
$\langle B, e\rangle$	
$\langle R, S\rangle$	
$\langle B, C+e\rangle$	
$\langle B+yf, C+\beta f\rangle \quad (y,\beta\in\mathbf{R})$	
$\langle B+e, cB+dC\rangle \quad ((c,d)\in \mathbf{R}^2\backslash\{0\})$	

3-dimensional subalgebras

$\langle R, S, F\rangle$
$\langle B, R, F\rangle$
$\langle aR+bS, e, f\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$
$\langle R+\beta f, S+\beta' f, e\rangle \quad (\beta,\beta'\in\mathbf{R})$
$\langle R+\beta f, B, e\rangle \quad (\beta\in\mathbf{R})$
$\langle B, e, f\rangle$
$\langle B, R+\alpha e+f, C+e\rangle \quad (\alpha\in\mathbf{R})$
$\langle aR+bS+e, B, C\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$
$\langle aR+bS+f, B, C\rangle \quad (a,b\in\mathbf{R})$
$\langle B+e, cB+dC, R+\alpha e+f\rangle \quad (\alpha\in\mathbf{R}, (c,d)\in \mathbf{R}^2\backslash\{0\})$
$\langle B, R+\beta f, e\rangle \quad (\beta\in\mathbf{R}, \beta\neq 1)$
$\langle B, dC+\beta f, e\rangle \quad ((d,\beta)\in \mathbf{R}^2\backslash\{0\})$
$\langle B, C, aR+bS\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$

4-dimensional subalgebras not containing $\mathcal{G}'$

$\langle R, S, e, f\rangle$
$\langle R, B, e, f\rangle$
$\langle R+\alpha e, S+\alpha' e, B, C\rangle \quad (\alpha,\alpha'\in\mathbf{R})$
$\langle aR+bS, B, C, f\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$
$\langle R+\beta f, S+\beta' f, B, C\rangle \quad (\beta,\beta'\in\mathbf{R})$
$\langle B, C, aR+bS, e\rangle \quad ((a,b)\in \mathbf{R}^2\backslash\{0\})$

5-dimensional subalgebras

Up to the action of Γ, there is only one 5-dimensional subalgebra not containing $\mathcal{G}'$, namely $\langle R, S, B, C, f\rangle$.

that π satisfies the assumptions of Section 2 and denote by N the Nilradical of Σ. Then $\dim N \geq 4$.

Assume that we are in the situation of the theorem but $\dim N < 4$. We will show that these assumptions lead to a contradiction.

LEMMA 1. *We have $\ell(\Sigma) \cong l_2^{\mathbf{C}} \times l_2$.*

Proof. Since $\dim N \leq 3$ we have $\ell(\Sigma) \cong l_2^{\mathbf{C}} \times l_2$ or $\ell(\Sigma) \cong l_2^3$. The latter possibility is excluded by Lemma 8 of Section 2.

Lemma 1 implies that $\Sigma = L_{2,n}^{\mathbf{C}} \times L_2$ for some $n \in \mathbf{N} \cup \{\infty\}$.

LEMMA 2. *Σ acts transitively on $\mathcal{P}\backslash W$, and $L_2' \subseteq \Sigma_{[v,W]}$. The invariant subgroup $(L_{2,n}^{\mathbf{C}})'$ has a trivial centralizer in Σ_o for each affine point $o \in \mathcal{P}\backslash W$.*

Proof. By Lemma 7 of Section 2 we know that $L_2' \subseteq \Sigma_{[v,W]}$. Let $o \in \mathcal{P}\backslash W$ be an affine point. If $C_{\Sigma_o}(L_{2,n}^{\mathbf{C}})' \neq 1$, then $(L_{2,n}^{\mathbf{C}})' \subseteq \Sigma_{[v,v]}$ by Lemma 5 of Section 2. But then Lemma 3 of Section 2 implies that $(L_{2,n}^{\mathbf{C}})' = \Sigma_{[v,W]}$, contradicting the fact that $L_2' \subseteq \Sigma_{[v,W]}$.

If Σ is not transitive on $\mathcal{P}\backslash W$, then there exists an affine point $o \in \mathcal{P}\backslash W$ with $\dim \Sigma_o \geq 3$. By an inspection of the list of subalgebras of $l_2^{\mathbf{C}} \times l_2$ given in Section 3, we see that $(L_{2,n}^{\mathbf{C}})' = \exp(\langle B, C\rangle)$ has a non-trivial centralizer in Σ_o; a contradiction.

LEMMA 3. *$\Sigma_{[v,W]} = L_2$ and $n = 1$, i.e. $\Sigma = L_2^{\mathbf{C}} \times L_2$.*

Proof. Choose an affine point $o \in \mathcal{P}\backslash W$. By Lemma 2 the stabilizer Σ_o is a 2-dimensional connected subgroup of Σ which contains no invariant subgroup of Σ, and the centralizer of $(L_{2,n}^{\mathbf{C}})'$ in Σ_o is trivial. By the list of 2-dimensional subalgebras of $l_2^{\mathbf{C}} \times l_2$ of Section 3 there are four possible cases for the subalgebra $\ell(\Sigma_o)$ (up to an automorphism of $l_2^{\mathbf{C}} \times l_2$). These cases are:

1. $\langle bS + f, R + b'S\rangle (b, b' \in \mathbf{R})$,
2. $\langle aR + bS, R + e\rangle ((a, b) \in \mathbf{R}^2\backslash\{0\})$,
3. $\langle S, R + f\rangle$,
4. $\langle R, S\rangle$.

In any case, $C_{\Sigma_o} L_2 \neq 1$ and Lemma 5 and Lemma 3 of Section 2 imply $L_2 = \Sigma_{[v,W]}$. By Lemma 6 of Section 2, we know that $Z(L_{2,n}^{\mathbf{C}}) \subseteq Z(\Sigma) \subseteq \Sigma_{[v,W]} = L_2$, hence $Z(L_{2,n}^{\mathbf{C}}) = 1$, i.e. $n = 1$.

Choose $o \in \mathcal{P}\backslash W$ and let $Y := o \vee v \in \mathcal{L}_v\backslash\{W\}$. We choose the coordinates in $\ell(\Sigma)$ in such a way that $\ell(\Sigma_o)$ is one of the four subalgebras in the proof of Lemma 3. If we let $C := C_{\Sigma_o} L_2$, then we have seen that $\dim C \geq 1$, and obviously $C \subseteq L_2^{\mathbf{C}} \cap \Sigma_{[Y]}$. The group $M := \exp\langle R, S\rangle$ is 2-dimensional abelian with $C \cap M \neq 1$, hence $M \subseteq \Sigma_Y$. This implies $\Sigma_Y = M \times L_2$ i.e. $\ell(\Sigma_Y) = \langle R, S, e, f\rangle$. We describe $L_2^{\mathbf{C}}$ in the coordinates of Section 3, i.e. $L_2^{\mathbf{C}} = \{(z, b) : z \in \mathbf{C}\backslash\{0\}, b \in \mathbf{C}\}$. In these coordinates we have $M = \{(z, 0) : z \in \mathbf{C}\backslash\{0\}\}$. The group L_2 is described similarly as $L_2 = \{(s,t) : s,t \in \mathbf{R}\}$ (with multiplication $(s_1,t_1)\cdot(s_2,t_2) = (s_1+s_2, e^{s_2}t_1+t_2)$).

LEMMA 4. *Σ acts transitively on $W\backslash\{v\}$ and $\mathcal{L}\backslash\mathcal{L}_v$.*

Proof. Assume to the contrary that there exists a 1-dimensional orbit $\mathcal{B}$ of Σ on $W\backslash\{v\}$. Since $C \subseteq \Sigma_{[Y]}$, there exists a $b \in \mathcal{B}$ with $C \not\subseteq \Sigma_b$, hence $L_2^{\mathbf{C}} \not\subseteq \Sigma_b$. By the list of 5-dimensional subalgebras, a 5-dimensional subalgebra of $l_2^{\mathbf{C}} \times l_2$ which does not contain the subalgebra $l_2^{\mathbf{C}}$ contains the commutator of $l_2^{\mathbf{C}} \times l_2$. This implies $\Sigma_b = \Sigma_{[\mathcal{B}]} \supseteq \Sigma'$. Since $L_2 = \Sigma_{[v,W]}$ is a regular normal subgroup of the action of Σ_b on $\mathcal{L}_b\backslash\{W\}$, we get $\Sigma_{[b]} \neq 1$. Since each element of Σ_b fixes $\mathcal{B}$ pointwise, we know $\Sigma_{[b]} = \Sigma_{[b,W]} \subseteq \Sigma_{[W,W]}$. Therefore the group $\Sigma_{[W,W]}$ must be abelian ([15, 8.1]), but it contains L_2; a contradiction. A dual application of Lemma 2 yields the last assertion.

Let $K := (L_2^{\mathbf{C}})' = \{(1,b) : b \in \mathbf{C}\}$ and $\Delta := K \times L_2 \trianglelefteq \Sigma$. The above calculation of Σ_Y implies that the group K acts regulary on $\mathcal{L}_v\backslash\{W\}$. Hence Δ acts regularly on $\mathcal{P}\backslash W$ and by duality Δ acts regularly on $\mathcal{L}\backslash\mathcal{L}_v$, too. Again by duality the stabilizers Σ_Y and Σ_b for an arbitrary point $b \in W\backslash\{v\}$ are conjugate in Σ, hence Σ_Y fixes a point $b \in W\backslash\{v\}$. Let $X := o \vee b \in \mathcal{L}\backslash\mathcal{L}_v$.

LEMMA 5. *For each line $A \in \mathcal{L}_v\backslash\{W\}$ there exists exactly one point $\phi(A) \in W\backslash\{v\}$ with $\Sigma_A = \Sigma_{\phi(A)}$ and the mapping $\phi : \mathcal{L}_v\backslash\{W\} \to W\backslash\{v\}$ is a bijection.*

Proof. We have proved $\Sigma_Y = \Sigma_b$ for some point $b \in W\backslash\{v\}$. Assume that there is another point $b' \in W\backslash\{v,b\}$ with $\Sigma_Y = \Sigma_{b'}$. Since $\Sigma_b = \Sigma_Y = M \times L_2$, we have $\Sigma_{[W]} = L_2$. The effective factor group $\Sigma/\Sigma_{[W]} \cong L_2^{\mathbf{C}}$ has the subgroup M as its stabilizer at the point b, i.e. the action of $\Sigma/\Sigma_{[W]}$ on $W\backslash\{v\}$ is equivalent to the natural action of $L_2^{\mathbf{C}}$ on $\mathbf{C}$. Hence $\Sigma_Y = \Sigma_{b,b'} = \Sigma_{[W]} = L_2$; a contradiction. Since Σ is transitive on $\mathcal{L}_v\backslash\{W\}$, this proves the first statement. By duality we get a map $\psi : W\backslash\{v\} \to \mathcal{L}_v\backslash\{W\}$ inverse to ϕ.

Coordinatize $\mathcal{P}\backslash W$ by Δ. Then

$$\mathcal{L}_v\backslash\{W\} = \{\{p\} \times L_2 : p \in K\}.$$

LEMMA 6. *$\Sigma_o \cong \mathbf{R} \times \mathrm{SO}(2)$ acts regularly on $\mathcal{L}_o\backslash\{Y, o \vee \phi(Y)\}$ and on $\mathcal{L}_v\backslash\{Y,W\}$.*

Proof. Let $Y' \in \mathcal{L}_v\backslash\{Y,W\}$. The group $\Sigma_{o,Y'}$ fixes the quadrangle $v, o, \phi(Y)$, $Y' \wedge (o \vee \phi(Y'))$, hence $\Sigma_{o,Y'} = 1$. Consequently Σ_o acts regularly on the cylinder $\mathcal{L}_v\backslash\{Y,W\}$ and in particular $\Sigma_o \cong \mathbf{R} \times \mathrm{SO}(2)$. The lemma on quadrangles and Lemma 5 imply the regularity of Σ_o on $\mathcal{L}_o\backslash\{Y, o \vee \phi(Y)\}$.

By our choice of the coordinate system in $l_2^{\mathbf{C}} \times l_2$ there are the following two possibilities for the subalgebra $\ell(\Sigma_o)$:

- Case A: $\ell(\Sigma_o) = \langle R + bf, S + df \rangle$ with $b, d \in \mathbf{R}$.
- Case B: $\ell(\Sigma_o) = \langle R + ae, S + ce \rangle$ with $a, c \in \mathbf{R}$.

CASE A: We get

$$\Sigma_o = \{(\mathrm{e}^z, 0, b\,\Re z + d\,\Im z, 0) : z \in \mathbf{C}\}$$

where $\Re z$ and $\Im z$ denote the real and imaginary part of $z \in \mathbf{C}$ respectively. Since Σ_o is isomorphic to $\mathbf{R} \times \mathrm{SO}(2)$ we get $d = 0$. Since Σ_o acts regularly on $\mathcal{L}_v \backslash \{Y, W\}$ the group Σ_o must act regularly on $X \backslash \{o, \phi(Y)\}$, too. The orbits of Σ_o on Δ which are not contained in L_2 are given by

$$\mathcal{B}_{s,t} = \{(1, \mathrm{e}^z, s, \mathrm{e}^{b\,\Re z} t) \colon\ z \in \mathbf{C}\} \quad \text{with} \quad s, t \in \mathbf{R}.$$

Choose $s, t \in \mathbf{R}$ with $X \backslash \{o, \phi(Y)\} = \mathcal{B}_{s,t}$. Since o is in the closure of $X \backslash \{o, \phi(Y)\}$ we get $s = 0$ and $t = 0$ or $b > 0$. If $t = 0$ then $X \backslash \{\phi(Y)\} = K \times \{1\}$. But the transitivity of Δ on $\mathcal{L} \backslash \mathcal{L}_v$ implies $\mathcal{L} \backslash \mathcal{L}_v = \{K \times \{q\} \colon\ q \in L_2\}$; a contradiction. Hence we get $b > 0 \neq t$. In this case:

$$\begin{aligned} X \backslash \{\phi(Y)\} &= \{(1, \mathrm{e}^z, 0, t\,\mathrm{e}^{b\,\Re z}) : z \in \mathbf{C}\} \cup \{(1,0,0,0)\} \\ &= \{(1, \mathrm{e}^z, 0, t|\mathrm{e}^z|^b) : z \in \mathbf{C}\} \cup \{(1,0,0,0)\} \\ &= \{(1, z, 0, t|z|^b) : z \in \mathbf{C}\}. \end{aligned}$$

For each $z \in \mathbf{C}$ we get the line

$$X(z) := X^{(1,z,0,-t|z|^b)} = \{(1, w, 0, t|w - z|^b - t|z|^b) : w \in \mathbf{C}\}.$$

Since Δ acts regularly on $\mathcal{L} \backslash \mathcal{L}_v$ these lines are pairwise distinct. All lines $X(z)$ pass through o and since the map $z \mapsto s(|1 - z|^b - |z|^b)$ is not one-to-one we get two complex numbers $z, z' \in \mathbf{C}$ with $z \neq z'$ such that the distinct lines $X(z), X(z')$ intersect in at least two points; a contradiction.

CASE B: In this case we have

$$\Sigma_o = \{(\mathrm{e}^z, 0, 0, a\,\Re z + c\,\Im z) : z \in \mathbf{C}\}.$$

Since $\Sigma_o \cong \mathbf{R} \times \mathrm{SO}(2)$ we get $c = 0$. The orbits of Σ_o on Δ which are not contained in L_2 are given by

$$\mathcal{B}_{s,t} = \{(1, \mathrm{e}^z, s, t + a(1 - \mathrm{e}^s)\,\Re z) : z \in \mathbf{C}\}.$$

As in Case A we can choose $s, t \in \mathbf{R}$ with $X \backslash \{o, \phi(Y)\} = \mathcal{B}_{s,t}$. The condition $o \in \overline{\mathcal{B}_{s,t}}$ implies in turn: $s = 0$ and $t = 0$. This shows $X \backslash \{\phi(Y)\} = K \times \{1\}$ and, as in Case A, this immediately implies a contradiction.

Now all cases have led to a contradiction and the theorem is proved.

References

1. Betten, D.: Komplexe Schiefparabelebenen, *Abh. Math. Sem. Univ. Hamburg* **48** (1979), 76–88.

2. Betten, D.: Zur Klassifikation 4-dimensionaler projektiver Ebenen, *Arch. Math.* **35** (1980), 187–192.
3. Betten, D. and Knarr, N.: Rotationsflächenebenen, *Abh. Math. Sem. Univ. Hamburg* **52** (1987), 227–234.
4. Betten, D.: 4-Dimensional compact projective planes with a 7-dimensional collineation group, *Geom. Dedicata* **36** (1990), 151–170.
5. Betten, D.: Orbits in 4-dimensional compact projective planes, *J. Geom.* **42** (1991), 30–40.
6. Betten, D.: 4-Dimensional compact projective planes with a nilpotent collineation group, *Mitt. Math. Ges. Hamburg* **12** (1991), 741–747.
7. Betten, D.: 4-Dimensional compact projective planes with a 5-dimensional nilradical (to appear).
8. Betten, D. and Klein, H.: 4-Dimensional compact projective planes with two fixed points (to appear).
9. Betten, D. and Polster, B.: 4-Dimensional compact projective planes of orbit type (1,1) (to appear).
10. Hilgert, J., Hofmann, K. H. and Lawson, J. D.: *Lie Groups, Convex Cones and Semigroups*, Oxford, 1989.
11. Knarr, N.: Topologische Differenzenflächenebenen, Diplomarbeit, Kiel, 1984.
12. Knarr, N.: Topologische Differenzenflächenebenen mit nichtkommutativer Standgruppe, Dissertation, Kiel, 1986.
13. Löwen, R.: Four-dimensional compact projective planes with a nonsolvable automorphism group, *Geom. Dedicata* **36** (1990), 225–234.
14. Mubarakzjanov, G. M.: On solvable Lie-algebras, *Izv. Vyssh. Uchebn. Zaved. Math.* **1** (1963), 114–123.
15. Pickert, G.: *Projektive Ebenen*, Springer, Berlin, 1975.
16. Salzmann, H. R.: Kollineationsgruppen kompakter 4-dimensionaler Ebenen, *Math. Z.* **117** (1970), 112–124.
17. Salzmann, H. R.: Kollineationsgruppen kompakter 4-dimensionaler Ebenen II, *Math. Z.* **121** (1971), 104–110.
18. Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R. and Stroppel, M.: *Compact Projective Planes*, de Gruyter, Berlin, 1995.