Algebraic Structures and Operator Calculus

Mathematics and Its Applications

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Algebraic Structures and Operator Calculus

Volume II:
Special Functions and Computer Science

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To our parents
In memoriam: Leo and Cecilia, Joseph and Anne

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Preface

In this volume we will present some applications of special functions in computer science. This largely consists of adaptations of articles that have appeared in the literature. Here they are presented in a format made accessible for the non-expert by providing some context. The material on group representations and Young tableaux is introductory in nature. However, the algebraic approach of Chapter 2 is original to the authors and has not appeared previously. Similarly, the material and approach based on Appell states, so formulated, is presented here for the first time.

As in all volumes of this series, this one is suitable for self-study by researchers. It is as well appropriate as a text for a course or advanced seminar.

The solutions are tackled with the help of various analytical techniques, such as generating functions, and probabilistic methods/insights appear regularly. An interesting feature is that, as has been the case in classical applications to physics, special functions arise — here in complexity analysis. And, as in physics, their appearance indicates an underlying Lie structure.

Our primary audience is applied mathematicians and theoretical computer scientists. We are quite sure that pure mathematicians will find this volume interesting and useful as well.

We expect this volume to have a utility between a reference and a monograph. We wish to make available in one volume a group of works and results scattered in the literature while providing some background to the mathematics involved which the reader will no doubt find appealing in its own right.

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INTRODUCTION

I. General remarks

In this volume, we present some applications of algebraic, analytic, and probabilistic methods to some problems in computer science. Special functions, notably orthogonal polynomials, and Bessel functions will appear.

In *Chapter 1*, we present the basic data structures we will be studying and introduce the ideas guiding the analysis. *Chapter* 2 presents applications of orthogonal polynomials and continued fractions to the analysis of dynamic data structures in the model introduced by J. Françon and D.E. Knuth, called Knuth's model. The approach here is original with this exposition. The underlying algebraic/analytic structures play a prominent rôle. *Chapter 3* presents some results involving Bessel functions and Lommel polynomials arising from a study of the behavior of the symbol table. Then, in another direction, theta functions come up as solutions of the heat equation on bounded domains, describing the limiting behavior of the evolution of stacks and the banker's algorithm. In *Chapter 4* we present some basic material on representations of finite groups, including Fourier transform. Then Fourier transform is considered in more detail for abelian groups, particularly cyclic groups. Next we look at Krawtchouk polynomials, which arise in a variety of ways, e.g., in the study of random walks. The concluding *Chapter 5* discusses representations of the symmetric group and connections with Young tableaux. Then, a variety of applications of Young tableaux are shown, including examples related to questions in parallel processing. Chapters 4 and 5 mainly include material from the literature that we find particularly interesting as well as some new approaches. It is hoped that the reader will find these chapters to be useful as background for further study as well as providing examples having effective illustrative and reference value.

Remark. The remainder of this introductory chapter provides some basic information that the reader may find useful, and is included so that the volume is fairly self-contained. Here one can find the notational conventions used in this volume; they are consistent with those of Volume 1. Many formulas and much information may be found in the Handbook by Abramowitz&Stegun [2], an overall very handy reference for the material of this chapter and for properties of the special functions used throughout this volume. Also, see Rainville[73].

II. Some notations

1. First, we recall the gamma and beta functions, given by the integrals

$$\begin{aligned} \Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ B(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt \end{aligned} \tag{2.1}$$

   for $\operatorname{Re} x$, $\operatorname{Re} y > 0$.

2. In hypergeometric functions, we use the standard Pochammer notation:

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

   where Γ denotes the gamma function. Thus, we have for binomial coefficients:

$$\binom{n}{k} = (-1)^k\,\frac{(-n)_k}{k!}$$

3. For hypergeometric functions

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{x^n}{n!}$$

   for integer $p, q \geq 0$. For example, we have

$$\begin{aligned} {}_2F_0\left(\begin{matrix} a, b \\ \text{---} \end{matrix}\,\middle|\, x\right) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!}\,x^n \\ &= 1 + ab\,x + a(a+1)b(b+1)\,x^2/2 + \cdots \end{aligned}$$

4. For Stirling numbers, we use $S_{n,k}$ and $s_{n,k}$ for *Stirling numbers* of the *first* and *second* kinds, respectively. They may be defined by the relations:

$$\begin{aligned} (x)_n &= \sum_{k=0}^{n} S_{n,k}x^k \\ x^n &= \sum_{k=0}^{n} s_{n,k}(-1)^{n-k}(x)_k \end{aligned} \tag{2.2}$$

   See Knuth[53], p. 65ff. (where, however, a different notation is used).

5. For finite sets E, we denote the cardinality of E by $|E|$ or #E.

III. Orthogonal polynomials and continued fractions

Orthogonal polynomials arise frequently as solutions to three-term recurrence relations. A standard reference on orthogonal polynomials is Szëgo[81]. For our work Chihara[10] is a useful reference as well.

3.1 ORTHOGONAL POLYNOMIALS

Given a bounded, nondecreasing function $F(x)$ on $\mathbf{R}$, we define a corresponding *Stieltjes measure* $dF(x)$. The *moments* of dF are defined by

$$\mu_n = \int_{-\infty}^{\infty} x^n \, dF(x)$$

We will take F to be normalized so that $\mu_0 = 1$. It is assumed that the moments exist for all $n \geq 0$. Integration with respect to dF, *mathematical expectation* or *expected value* , is denoted by angle brackets $\langle \cdot \rangle$

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) \, dF(x) \tag{3.1.1}$$

in particular, we can write the moments as $\mu_n = \langle X^n \rangle$. In the *discrete* case, we have a probability sequence (discrete distribution), $\{ p_k \}_{k \geq 0}$, satisfying $p_k \geq 0$, $\sum p_k = 1$, with corresponding moments

$$\mu_n = \sum_{k=0}^{\infty} (a_k)^n \, p_k$$

where a_k denotes the value taken on with corresponding probability p_k. In the (absolutely) *continuous case*, we have a nonnegative density function $p(x)$, and

$$\mu_n = \int_{-\infty}^{\infty} x^n \, p(x) \, dx$$

We define an inner product $\langle \cdot, \cdot \rangle$ on the set of polynomials, which is the mathematical expectation of their product. On the basis $\{ x^n \}$, this is given in terms of the moments:

$$\langle x^n, x^m \rangle = \langle X^n \cdot X^m \rangle = \int_{-\infty}^{\infty} x^n \cdot x^m \, dF(x) = \mu_{n+m}$$

The corresponding sequence of orthogonal polynomials, $\{ \phi_n(x) \}_{n \geq 0}$, say, may be determined by applying the Gram-Schmidt orthogonalization procedure to the basis $\{ x^n \}$. The polynomials $\{ \phi_n(x) \}$ satisfy

$$\langle \phi_n, \phi_m \rangle = \langle \phi_n(X) \, \phi_m(X) \rangle = \int_{-\infty}^{\infty} \phi_n(x) \phi_m(x) \, dF(x) = \begin{cases} \gamma_n, & n = m \\ 0, & n \neq m \end{cases} \tag{3.1.2}$$

where γ_n are the *squared norms*. We normalize γ_0 to 1.

With $\phi_n(x) = x^n + \cdots$, monic polynomials, the $\{\phi_n(x)\}$ satisfy a three-term recurrence relation of the form:

$$x\phi_n = \phi_{n+1} + a_n\phi_n + b_n\phi_{n-1} \tag{3.1.3}$$

with $\phi_0 = 1$, $\phi_1 = x - a_0$. Orthogonality, comparing $\langle x\phi_n, \phi_{n-1}\rangle = b_n\gamma_{n-1}$ with $\langle \phi_n, x\phi_{n-1}\rangle = \gamma_n$, yields the relation:

$$\gamma_n = b_1 b_2 \cdots b_n \tag{3.1.4}$$

3.2 CONTINUED FRACTIONS

A general reference for continued fractions is Jones&Thron[47]. Two sequences of (real or complex) numbers $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 0}$ determine a continued fraction:

$$b_0 + a_1\Big/b_1 + a_2\Big/b_2 + \ldots + a_n\Big/b_n + \ldots$$

which is the limit as $n \to \infty$ of the *approximants*:

$$\frac{P_n}{Q_n} = b_0 + a_1\Big/b_1 + a_2\Big/b_2 + \ldots + a_n\Big/b_n$$

P_n and Q_n are the n^{th} *partial numerators* and *partial denominators*, respectively. They satisfy the recurrence relations:

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{3.2.1}$$

with initial conditions

$$P_{-1} = 1,\ P_0 = b_0, \qquad Q_{-1} = 0,\ Q_0 = 1$$

respectively. One may remark that this setup corresponds to the matrix relations:

$$\begin{pmatrix} Q_{n-1} & P_{n-1} \\ Q_n & P_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_n & b_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ a_1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_0 \end{pmatrix}$$

3.2.1 Connection with orthogonal polynomials

Consider the continued fraction

$$1\big/1 - a_0x - b_1x^2\big/1 - a_1x - b_2x^2\big/\ldots\big/1 - a_{n-1}x - b_nx^2\big/\ldots \tag{3.2.1.1}$$

From equation (3.2.1), the partial numerators and denominators satisfy

$$u_{n+1} = (1 - a_nx)u_n - b_nx^2u_{n-1}$$

for $n \geq 1$, where the *numerator polynomials*, $P_n(x)$, satisfy the initial conditions $P_0 = 0$, $P_1 = 1$, and the *denominator polynomials*, $Q_n(x)$, satisfy $Q_0 = 1$, $Q_1 = 1 - a_0x$.

The reciprocal polynomials, $\psi_n(x)$, defined as

$$\psi_n(x) = x^n\, Q_n(1/x)$$

satisfy the three-term recurrence

$$x\psi_n = \psi_{n+1} + a_n\psi_n + b_n\psi_{n-1} \tag{3.2.1.2}$$

with initial conditions $\psi_0 = 1$, $\psi_1 = x - a_0$. These give a sequence of orthogonal polynomials with squared norms $\gamma_n = b_1b_2\cdots b_n$, $n \geq 0$.

IV. Bessel functions, Lommel polynomials, theta functions

4.1 BESSEL FUNCTIONS AND LOMMEL POLYNOMIALS

The reader is referred to G.N. Watson's [84] treatise for an extensive presentation of the theory of Bessel functions. Here we indicate the features that are important for our applications.

The Bessel function J_ν is defined by :

$$J_\nu(z) = (z/2)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\nu+1)} \tag{4.1.1}$$

This function arises as the solution of the differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0 \tag{4.1.2}$$

In the present study, a principal feature of Bessel functions is the fundamental recurrence:

$$2\nu z^{-1}J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z) \tag{4.1.3}$$

Iterating this recurrence, *Lommel polynomials* come up as the coefficients expressing $J_{\nu+n}(z)$ as a linear combination of $J_\nu(z)$ and $J_{\nu-1}(z)$:

$$J_{\nu+n}(z) = J_\nu(z)R_{n,\nu}(z) - J_{\nu-1}(z)R_{n-1,\nu+1}(z) \tag{4.1.4}$$

This holds for nonnegative integer n. For $n = 1$, we have $R_{1,\nu}(z) = 2\nu/z$, while $R_{0,\nu}(z) = 1$. The Lommel polynomials satisfy the recurrence

$$2z^{-1}(n+\nu)\,R_n = R_{n+1} + R_{n-1}$$

with the initial conditions $R_{-1} = 0$, $R_0 = 1$. The R_n may be given explicitly in the form

$$R_{n,\nu}(z) = \sum_{k=0}^{[n/2]} \binom{n-k}{k}(-1)^k \frac{(\nu)_{n-k}}{(\nu)_k}\left(\frac{2}{z}\right)^{n-2k}$$

They are used to solve the recurrence

$$x\psi_n = \psi_{n+1} + \varepsilon^{-1} n\psi_n + \psi_{n-1} \tag{4.1.5}$$

which has as general solution

$$\psi_n = c_1 R_{n,-\varepsilon x}(-2\varepsilon) - c_2 R_{n-1,1-\varepsilon x}(-2\varepsilon)$$

i.e., with initial values $\psi_{-1} = c_2$, $\psi_0 = c_1$. Eq. (4.1.4) can be interpreted as expressing the solution to $x\psi_n = \psi_{n+1} + \varepsilon^{-1}n\psi_n + \psi_{n-1}$ in the form

$$J_{n-\varepsilon x}(-2\varepsilon) = J_{-\varepsilon x}(-2\varepsilon)R_{n,-\varepsilon x}(-2\varepsilon) - J_{-1-\varepsilon x}(-2\varepsilon)R_{n-1,1-\varepsilon x}(-2\varepsilon) \tag{4.1.6}$$

which combines features of both eqs. (4.1.3), (4.1.4).

4.2 THETA FUNCTIONS

The four Jacobi *theta functions* are given by

$$\vartheta_1(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z$$

$$\vartheta_2(z,q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \cos(2n+1)z$$

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} \cos 2nz$$

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

V. Some analytic techniques

Here we indicate some basic analytic techniques that will be used. Included are: asymptotic behavior of coefficients of a power series, central limit approximation of sums involving binomial coefficients, and Lagrange inversion.

5.1 BASICS OF ASYMPTOTICS

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in a neighborhood of 0 in the complex plane. We are interested in the behavior of $|a_n|$ as $n \to \infty$. The easiest observation is:

5.1.1 Proposition. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in a disk of radius $R > 1$. Then the coefficients a_n decrease exponentially fast.*

Proof: The terms of the series $\sum_{n=0}^{\infty} |a_n|(R-\epsilon)^n$ converge to zero for any $\epsilon > 0$. Choose ϵ so that $R - \epsilon > 1$. ■

Because of this, one looks for the singularities of functions when it is known that the coefficients a_n grow as $n \to \infty$. For our purposes, the following suffices.

5.1.2 Lemma. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be meromorphic, analytic in a neighborhood of 0 in the complex plane. Let ζ denote the pole of f nearest to the origin. If ζ is a pole of order p, then we have the estimate*

$$|a_n| \sim \frac{A}{\Gamma(p)} \frac{n^{p-1}}{\zeta^{n+p}}$$

where $A = \lim_{z\to\zeta} f(z)(z-\zeta)^p$.

Proof: Write $g(z) = f(z) \cdot (z-\zeta)^p$. So $A = g(\zeta)$ and

$$f(z) = \frac{A}{(z-\zeta)^p} + \frac{g(z) - g(\zeta)}{(z-\zeta)^p}$$

where the second term has at most a pole of order $< p$. Now, for $|z| < |\zeta|$, the binomial theorem implies

$$\frac{A}{(z-\zeta)^p} = (-1)^p A \sum_{n=0}^{\infty} \frac{(n+1)_{p-1}}{\Gamma(p)} \frac{z^n}{\zeta^{n+p}}$$

At a pole of larger modulus, the same estimate gives a rate of growth exponentially slower, whereas terms of lower order poles grow at correspondingly lower powers of n. ■

See Flajolet-Vitter[28] for more along these lines.

5.2 CENTRAL LIMIT APPROXIMATION

The central limit approximation is a detailed application of the theorem of DeMoivre-Laplace that one can approximate sums involving binomial coefficients, interpreted as taking expectations with respect to binomial distributions, by Gaussian integrals. Here is a useful version.

5.2.1 Lemma. *Let f be a polynomial. Then, as $N \to \infty$,*

$$\sum_{|k|<N/2} f(k)\binom{N}{k+N/2} \sim \int_{-\infty}^{\infty} f(\alpha)e^{-2\alpha^2}\,d\alpha$$

Another formulation of the same statement is:

$$\binom{N}{k+N/2} \sim \binom{N}{N/2} e^{-2\alpha^2}\sqrt{N}\,d\alpha$$

where $k = \alpha\sqrt{N}$. In particular, we have

5.2.2 Lemma. *For $p > 0$, as $N \to \infty$,*

$$\sum_{|k|<N/2} k^{2p}\binom{N}{k+N/2} \sim \binom{N}{N/2} N^{p+1/2}\sqrt{\frac{\pi}{2}}\frac{1}{2^p}\left(\frac{1}{2}\right)_p$$

Proof: This follows from the above remarks and the Gaussian integral

$$\begin{aligned}\int_{-\infty}^{\infty} \alpha^{2p}e^{-2\alpha^2}\,d\alpha &= 2\int_0^{\infty} \alpha^{2p}e^{-2\alpha^2}\,d\alpha \\ &= \frac{1}{2^{p+1/2}}\int_0^{\infty} t^{p-1/2}e^{-t}\,dt \\ &= \frac{1}{2^{p+1/2}}\Gamma(p+1/2)\end{aligned}$$

via the substitution $t = 2\alpha^2$. The result follows, using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. ∎

See Canfield[9] for more information on this topic.

5.3 LAGRANGE INVERSION

Given a function, f, analytic in a neighborhood of x_0, that is locally one-to-one, write $y = f(x)$. Lagrange inversion provides a way of finding expansions of functions of x in terms of the variable y. In particular, the relation $x = f^{-1}(y)$ is the inverse of f. The idea is that we want the expansion in terms of y of any function $g(x)$ analytic in a neighborhood of x_0.

5.3.1 Proposition. *Let $y = f(x)$ be analytic near x_0, $y_0 = f(x_0)$, with $f'(x_0) \neq 0$. For g analytic near x_0, the Lagrange inversion formula gives the expansion*

$$g(x) = g(x_0) + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!} \left(\frac{d}{dx}\right)^{k-1} \left[g'(x) \left\{ \frac{x - x_0}{f(x) - y_0} \right\}^k \right]_{x=x_0}$$

See Sansone-Gerretsen[79] for details. The proof uses Cauchy's integral formula and integration by parts.

VI. Polynomials: reference information

Here we include some basic reference material on the classes of orthogonal polynomials that come up, notably in the study of dynamic data structures. In general, references are Szëgo[81] and Chihara[10].

First we note

1. The inner product is denoted by angle brackets $\langle \cdot, \cdot \rangle$, thus

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dF(x)$$

   In particular, for an orthogonal sequence $\{ \phi_n \}$, we have

$$\langle \phi_n, \phi_m \rangle = \int_{-\infty}^{\infty} \phi_n(x) \phi_m(x)\, dF(x) = \gamma_n \, \delta_{nm}$$

   where δ_{nm} is Kronecker's delta equal to 1 if $n = m$, 0 otherwise, and γ_n are the squared norms.
2. The *moment generating function* denotes a generating function for the sequence of moments $\{ \mu_n \}$. Note that for symmetric distributions, where the odd moments vanish, we often write the generating function for $\{ \mu_{2n} \}$.

6.1 POLYNOMIALS CORRESPONDING TO DISCRETE DISTRIBUTIONS

Here we list basic information on polynomials orthogonal with respect to the *binomial* and *Poisson* distributions, respectively.

- Binomial distribution — Krawtchouk polynomials $\{ K_n(x) \}$

Recurrence formula

$$x\, K_n = K_{n+1} + n(N - n + 1) K_{n-1}$$

with $K_0 = 1$, $K_1 = x$.

Measure of orthogonality and squared norms

$$\langle K_n, K_m \rangle = 2^{-N} \sum_{k=0}^{N} \binom{N}{k} K_n(2k-N) K_m(2k-N) = \gamma_n \, \delta_{nm}$$

$$\gamma_n = n! \, (-1)^n (-N)_n$$

Moment generating function

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \, \mu_n = (\cosh s)^N$$

Generating function for the polynomials

$$\sum_{n=0}^{N} \frac{v^n}{n!} K_n(x) = (1+v)^{(N+x)/2} (1-v)^{(N-x)/2}$$

Formula for the Krawtchouk polynomials

$$K_n(x) = n! \sum_{k=0}^{n} \binom{(N+x)/2}{k} \binom{(N-x)/2}{n-k} (-1)^{n-k}$$

- Poisson distribution — Poisson-Charlier polynomials $\{ P_n(x,t) \}$

Recurrence formula

$$x \, P_n = P_{n+1} + (t+n) P_n + tn P_{n-1}$$

with $P_0 = 1$, $P_1 = x - t$.

Measure of orthogonality and squared norms

$$\langle P_n, P_m \rangle = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} P_n(k) P_m(k) = \gamma_n \, \delta_{nm}$$

$$\gamma_n = n! \, t^n$$

Moments

$$\mu_n = \sum_k s_{n,k} \, t^k$$

Moment generating function

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \, \mu_n = \exp\left(t(e^s - 1)\right)$$

Generating function for the polynomials

$$\sum_{n=0}^{\infty} \frac{v^n}{n!} P_n(x,t) = (1+v)^x e^{-vt}$$

Formula for the Poisson-Charlier polynomials

$$P_n(x,t) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} t^{n-k} (-x)_k$$

Remark. The moments for the Poisson distribution are readily calculated by considering

$$\int_0^\infty y^x \, dF(x) = e^{-t} \sum_{k=0}^{\infty} \frac{y^k t^k}{k!} = e^{yt-t}$$

Differentiating n times with respect to y, then setting $y = 1$, yields

$$\int_0^\infty x(x-1)\cdots(x-n+1)\, dF(x) = \int_0^\infty (-1)^n (-x)_n \, dF(x) = t^n$$

Then the Stirling numbers of the second kind, eq. (2.2), convert these to the usual moments given above.

6.2 POLYNOMIALS CORRESPONDING TO CONTINUOUS DISTRIBUTIONS

Here we have Tchebychev and Hermite polynomials. Recall that the Tchebychev polynomials are determined according to

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta$$

The Hermite polynomials are orthogonal with respect to the *Gaussian* distribution.

- Tchebychev polynomials of the first kind $\{ T_n(x) \}$

Recurrence formula

$$2x\, T_n = T_{n+1} + T_{n-1}$$

with $T_0 = 1$, $T_1 = x$.

Measure of orthogonality and squared norms

$$\langle T_n, T_m \rangle = \frac{1}{\pi} \int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \gamma_n \, \delta_{nm}$$

$$\gamma_n = \frac{1}{2}, \quad n \geq 1$$

Moments

$$\mu_{2n} = (\tfrac{1}{2})_n/n! = \frac{1}{2^{2n}}\binom{2n}{n}$$

Moment generating function

$$\sum_{n=0}^{\infty} s^{2n}\,\mu_{2n} = \frac{1}{\sqrt{1-s^2}}$$

Generating function for the polynomials

$$\sum_{n=0}^{\infty} v^n\,T_n(x) = \frac{1-xv}{1-2xv+v^2}$$

Formula for the Tchebychev polynomials of the first kind

$$T_n(x) = \tfrac{1}{2}\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{n}{n-k}(-1)^k(2x)^{n-2k}$$

- Tchebychev polynomials of the second kind $\{\,U_n(x)\,\}$

Recurrence formula

$$2x\,U_n = U_{n+1} + U_{n-1}$$

with $U_0 = 1$, $U_1 = 2x$.

Measure of orthogonality and squared norms

$$\langle U_n, U_m\rangle = \frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,dx = \gamma_n\,\delta_{nm}$$

$$\gamma_n = 1\,, \quad n \geq 0$$

Moments

$$\mu_{2n} = (\tfrac{1}{2})_n/(n+1)! = \frac{1}{2^{2n}(n+1)}\binom{2n}{n}$$

Moment generating function

$$\sum_{n=0}^{\infty} s^{2n}\,\mu_{2n} = 2\,\frac{1-\sqrt{1-s^2}}{s^2}$$

Generating function for the polynomials

$$\sum_{n=0}^{\infty} v^n\,U_n(x) = \frac{1}{1-2xv+v^2}$$

Formula for the Tchebychev polynomials of the second kind

$$U_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2x)^{n-2k}$$

- Gaussian distribution — Hermite polynomials $\{ H_n(x,t) \}$

Recurrence formula

$$x\, H_n = H_{n+1} + tnH_{n-1}$$

with $H_0 = 1$, $H_1 = x$.

Measure of orthogonality and squared norms

$$\langle H_n, H_m \rangle = \int_{-\infty}^{\infty} H_n(x) H_m(x)\, e^{-x^2/2t}\, dx/\sqrt{2\pi t} = \gamma_n\, \delta_{nm}$$

$$\gamma_n = n!\, t^n$$

Moments

$$\mu_{2n} = (2t)^n \left(\tfrac{1}{2}\right)_n = \frac{(2n)!}{2^n\, n!}\, t^n$$

Moment generating function

$$\sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!}\, \mu_{2n} = e^{s^2 t/2}$$

Generating function for the polynomials

$$\sum_{n=0}^{\infty} \frac{v^n}{n!}\, H_n(x,t) = e^{vx - v^2 t/2}$$

Formula for the Hermite polynomials

$$H_n(x,t) = \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\tfrac{1}{2}\right)_k (-1)^k x^{n-2k} (2t)^k$$