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# Compactness of anisotropic arithmetic quotients

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These notes give the beginning of a treatment of reduction theory for classical groups following Tamagawa-Mostow and Godement's Bourbaki article. For the moment, the non-compact case is neglected.

- Affine heights
- Minkowski reduction
- Imbeddings of arithmetic quotients
- Mahler's criterion for compactness
- Compactness of anisotropic quotients of orthogonal groups

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## 1. Affine heights

Let  $K_v$  be the standard compact subgroup of  $GL(n, \mathbb{Q}_v)$ : namely, for  $\mathbb{Q}_v \approx \mathbb{R}$  the usual orthogonal group  $O(n)$ , and for  $\mathbb{Q}_v$  non-archimedean it is  $GL(n, \mathbb{Z}_v)$ . (The fact that these subgroups are *maximal* compact will not be needed.) Let  $V = \mathbb{Q}^n$ , and  $V_{\mathbb{A}} = V \otimes \mathbb{A}$ . Let  $GL(n, \mathbb{A})$  act on the right on  $\mathbb{A}^n$  by matrix multiplication.

For the *real* prime  $v$  of  $\mathbb{Q}$  define the **local height** function  $\eta_v$  on  $x = (x_1, \dots, x_n) \in V_{\mathbb{Q}_v} = \mathbb{Q}_v^n$  by

$$\eta_v(x) = \sqrt{x_1^2 + \dots + x_n^2}$$

For a *non-archimedean* prime  $v$  of  $\mathbb{Q}$  define the **local height** function  $\eta_v$  on  $x = (x_1, \dots, x_n) \in V_{\mathbb{Q}_v} = \mathbb{Q}_v^n$  by

$$\eta_v(x) = \sup_i |x_i|_v$$

A vector  $x \in V_{\mathbb{A}}$  is **primitive** if it is of the form  $x_o g$  where  $g \in GL(n, \mathbb{A})$  and  $x_o \in V_{\mathbb{Q}}$ . That is, it is an image of a *rational* point of the vectorspace by an element of the *adele* group. For  $x = (x_1, \dots, x_n) \in V_{\mathbb{Q}}$ , at almost all non-archimedean primes  $v$  the  $x_i$ 's are in  $\mathbb{Z}_v$  and have greatest common divisor 1 (locally). Since elements of the adele group are in  $K_v$  almost everywhere, this property is not changed by multiplication by  $g \in GL(n, \mathbb{A})$ . That is, any primitive vector  $x$  has the property that at almost all  $v$  the components of  $x$  are locally integral and have (local) greatest common divisor 1.

For primitive  $x \in V_{\mathbb{A}}$  define the **global height**

$$\eta(x) = \prod_v \eta_v(x_v)$$

Since  $x$  is primitive, at almost all finite primes the local height is 1, so this product has only finitely many non-1 factors.

- For  $t \in \mathbf{J}$  and primitive  $x \in \mathbb{A}^n$ ,  $\eta(tx) = |t|\eta(x)$ , where  $|t|$  is the idele norm.
- If a sequence of vectors in  $\mathbb{A}^n$  goes to 0, then their heights go to zero also.
- If the heights of some (primitive) vectors  $x_i$  go to zero, then there are scalars  $t_i \in \mathbb{Q}^\times$  so that  $t_i x_i$  goes to 0 in  $\mathbb{A}^n$ .

- For  $g \in GL(n, \mathbb{A})$  and  $c > 0$ , the set of non-zero vectors  $x \in \mathbb{Q}^n$  so that  $\eta(xg) < c$  is finite modulo  $\mathbb{Q}^\times$ . In particular, the infimum of  $\{\eta(xg) : x \in \mathbb{Q}^n - 0\}$  is positive, and is assumed.
- For a compact subset  $E$  of  $GL(n, \mathbb{A})$  there are constants  $c, c' > 0$  so that for all primitive vectors  $x$  and for all  $g \in E$

$$c\eta(x) \leq \eta(xg) \leq c'\eta(x)$$

*Proof:* The first assertion follows from the product formula.

For the *second* assertion: if a sequence of vectors  $x_i$  goes to 0, then for every large  $N > 0$  and small  $\varepsilon > 0$  there is  $i_o$  so that  $i \geq i_o$  implies  $\eta_v(x_v) < \varepsilon$  at archimedean primes, and  $x_v \in N\mathbb{Z}_v^n$  for every finite  $v$ . Then  $\eta(x) \leq \varepsilon^\ell/N$  where  $\ell$  is the number of archimedean primes. So the heights go to zero.

For the *third* assertion: suppose that  $\eta(x_i)$  goes to 0, for some primitive vectors  $x_i$ . At almost all finite  $v$  the vector  $x_i$  is in  $\mathbb{Z}_v^n$  and the entries have local gcd 1. Since  $\mathbb{Z}$  is a principal ideal domain, we can choose  $s_i \in \mathbb{Q}$  to that at *every* finite prime  $v$  the components of  $s_i x_i$  are locally integral and have greatest common divisor 1. Then the local contribution to the height function from *all* finite primes is 1. Therefore, the archimedean height of  $s_i x_i$ , Euclidean distance, goes to 0. Finally, we need some choice of trick to make the vectors go to 0 in  $\mathbb{A}^n$ . For example, for each index  $i$  let  $N_i$  be the greatest integer so that

$$\eta_\infty(s_i x_i) < \frac{1}{(N_i!)^2}$$

Let  $t_i = s_i \cdot N_i!$ . Then  $t_i x_i$  goes to 0 in  $\mathbb{A}^n$ .

For the *fourth* assertion: fix  $g \in GL(n, \mathbb{A})$ . Since  $K$  preserves heights, via the Iwasawa decomposition we may suppose that  $g$  is in the group  $P_{\mathbb{A}}$  of upper triangular matrices in  $GL(n, \mathbb{A})$ . Let  $g_{ij}$  be the  $(i, j)^{th}$  entry of  $g$ . Choose representatives  $x = (x_1, \dots, x_n)$  for non-zero vectors in  $\mathbb{Q}^n$  modulo  $\mathbb{Q}^\times$  such that, letting  $\mu$  be the first index with  $x_\mu \neq 0$ , then  $x_\mu = 1$ . That is,  $x$  is of the form

$$x = (0, \dots, 0, 1, x_{\mu+1}, \dots, x_n)$$

To illustrate the idea of the argument with a light notation, first consider  $n = 2$ , let  $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  and  $x = (1, y)$ . Thus,

$$x \cdot g = (1, y) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, b + yd)$$

From the definition of the local heights, at each place  $v$  of  $k$

$$\max(|a|_v, |b + yd|_v) \leq h_v(xg)$$

from which

$$|b + yd|_v \prod_{w \neq v} |a|_w \leq \prod_{\text{all } w} h_w(xg) = h(xg)$$

Since  $g$  is fixed,  $a$  is fixed, and at almost all places  $|a|_w = 1$ . Thus, for  $h(xg) < c$  there is a uniform  $c'$  such that

$$|b + yd|_v \leq c' \quad (\text{for all } v)$$

Since for almost all  $v$  the residue class field cardinality  $q_v$  is strictly greater than  $c$

$$|b + yd|_v \leq 1 \quad (\text{for almost all } v)$$

Therefore,  $b + yd$  lies in a compact subset  $C$  of  $\mathbb{A}$ . Since  $b, d$  are fixed, and since  $\mathbb{Q}$  is discrete and closed in  $\mathbb{A}$ , the collection of images  $\{b + dy : y \in k\}$  is discrete in  $\mathbb{A}$ . Thus, the collection of  $y$  such that  $b + dy$  lies in  $C$  is finite.

Now consider general  $n$  and  $x \in \mathbb{Q}^n$  such that  $h(xg) < c$ . Let  $\mu - 1$  be the least index such that  $x_\mu \neq 0$ . Adjuts by  $k^\times$  such that  $x_\mu = 1$ . For each  $v$ , from  $h(xg) < c$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \leq h(xg) < c$$

For almost all places  $v$  we have  $|g_{\mu-1,\mu-1}|_v = 1$ , so there is a uniform  $c'$  such that

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v < c' \quad (\text{for all } v)$$

For almost all  $v$  the residue field cardinality  $q_v$  is strictly greater than  $c'$ , so for almost all  $v$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \leq 1$$

Therefore,  $g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}$  lies in a compact subset  $C$  of  $\mathbb{A}$ . Since  $\mathbb{Q}$  is discrete, the collection of  $x_\mu$  is finite.

Continuing similarly, there are only finitely many choices for the other entries of  $x$ . Inductively, suppose  $x_i = 0$  for  $i < \mu - 1$ , and  $x_\mu, \dots, x_{\nu-1}$  fixed, and show that  $x_\nu$  has only finitely many possibilities. Looking at the  $\nu^{\text{th}}$  component  $(xg)_\nu$  of  $xg$ ,

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \dots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \leq h(xg) \leq c$$

For almost all places  $v$  we have  $|g_{\mu-1,\mu-1}|_w = 1$ , so there is a uniform  $c'$  such that for all  $v$

$$|(xg)_\nu|_v = |g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \dots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v < c'$$

For almost all  $v$  the residue field cardinality  $q_v$  is strictly greater than  $c'$ , so for almost all  $v$

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \dots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \leq 1$$

Therefore,

$$g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \dots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}$$

lies in the intersection of a compact subset  $C$  of  $\mathbb{A}$  with a closed discrete set, so lies in a finite set. Thus, the number of possibilities for  $x_\nu$  is finite. By induction we obtain the finiteness.

For the *last* assertion: let  $E$  be a compact subset of  $GL(n, \mathbb{A})$ , and let  $K = \prod_v K_v$ . Then  $K \cdot E \cdot K$  is compact, being the continuous image of a compact set. So without loss of generality  $E$  is left and right  $K$ -stable. By Cartan decompositions the compact set  $E$  of  $GL(n, \mathbb{A})$  is contained in a set

$$K \Delta K$$

where  $\Delta$  is a compact set of diagonal matrices in  $GL(n, \mathbb{A})$ . Let  $g = \theta_1 \delta \theta_2$  with  $\theta_i \in K$ , and  $x$  a primitive vector. By the  $K$ -invariance of the height,

$$\frac{\eta(xg)}{\eta(x)} = \frac{\eta(x\theta_1 \delta \theta_2)}{\eta(x)} = \frac{\eta(x\theta_1 \delta)}{\theta(x)} = \frac{\eta((x\theta_1)\delta)}{\eta((x\theta))}$$

Thus, the set of ratios  $\eta(xg)/\eta(x)$  for  $g$  in a compact set and  $x$  ranging over primitive vectors is exactly the set of values  $\eta(x\delta)/\eta(x)$  where  $\delta$  ranges over a compact set and  $x$  varies over primitives. With diagonal entries  $\delta_i$  of  $\delta$ ,

$$0 < \inf_{\delta \in \Delta} \inf_i |\delta_i| \leq \eta(x\delta)/\eta(x) \leq \sup_{\delta \in \Delta} \sup_i |\delta_i| < \infty$$

by compactness of  $\Delta$ . ///

## 2. Minkowski reduction

The previous preparations set things up to prove the basic reduction-theory result for non-compact quotients: we prove that there is a nice **approximate fundamental domain** for the action of  $GL(n, \mathbb{Q})$  on  $GL(n, \mathbb{A})$ .

**[2.0.1] Theorem:** (*Adelic form of Minkowski reduction*) Given  $g \in GL(n, \mathbb{A})$ , there are  $\gamma \in GL(n, \mathbb{Q})$  and  $\theta \in K$  so that

$$p = \gamma g \theta$$

is **upper-triangular** and so that the diagonal entries  $p_{ii}$  of  $p$  satisfy the **inequalities**

$$\left| \frac{p_{ii}}{p_{i+1, i+1}} \right| \geq \frac{\sqrt{3}}{2} \quad (\text{idele norm})$$

Further, for  $i < j$ , the entry  $p_{ij}$  of  $p$  can be arranged to lie in any specified set of representatives in  $\mathbb{A}$  for the quotient  $p_{ii}\mathbb{Q} \backslash \mathbb{A}$ , such as  $\mathbb{R}/\mathbb{Z} \times \widehat{\mathbb{Z}}$ .

**[2.0.2] Remark:** Combined with Strong Approximation for  $SL(n)$ , this recovers classical Minkowski reduction for  $SL(n, \mathbb{Z})$  on  $SL(n, \mathbb{R})$ . More importantly, it begins the general fundamental domain results, in terms of **Siegel sets**.

From above, given  $g \in GL(n, \mathbb{A})$  there is  $x \in \mathbb{Q}^n - 0$  such that  $\eta(xg) > 0$  is minimal among values  $\eta(x'g)$  with  $x \in \mathbb{Q}^n - 0$ . Take  $\gamma \in GL(n, \mathbb{Q})$  so that  $e_n \gamma = x$ , where  $\{e_i\}$  is the standard basis for  $\mathbb{Q}^n$ . By Iwasawa, there is  $\theta \in K$  such that  $p = \gamma g \theta$  is upper-triangular. Then

$$\eta(\gamma g \theta) = |p_{nn}| \quad (p_{ij} \text{ is } ij^{\text{th}} \text{ entry of } p)$$

Let  $H$  be the subgroup of  $GL(n, \mathbb{A})$  fixing  $e_n$  and stabilizing the subspace spanned by  $e_1, \dots, e_{n-1}$ . Then  $H \approx GL(n-1, \mathbb{A})$ , and by induction we can suppose that  $|p_i/p_{i+1, i+1}| \geq \frac{\sqrt{3}}{2}$  already for  $i < n-1$ . Looking at just the lower-right two-by-two block inside these  $n$ -by- $n$  matrices, it suffices to consider  $n = 2$ .

Repeating: given  $g \in GL(2, \mathbb{A})$  there is  $x \in \mathbb{Q}^2 - 0$  such that  $\eta(xg)$  is positive and minimal among all the values  $\eta(x'g)$  with  $x \in \mathbb{Q}^2 - 0$ . Take  $\gamma \in GL(2, \mathbb{Q})$  such that  $(0 \ 1)\gamma = x$ . By Iwasawa there is  $\theta$  in the standard compact subgroup  $K$  of  $GL(2, \mathbb{A})$  such that  $p = \gamma g \theta$  is upper-triangular, say

$$p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

We wish to see that the minimality of  $\eta(xg) = \eta((0 \ 1)p)$  gives  $|a/d| \geq \frac{\sqrt{3}}{2}$ . Let  $x' = (1, t) \in \mathbb{Q}^2$ . The inequality

$$\eta((0 \ 1)p) \leq \eta(x'p)$$

gives

$$|d| \leq \eta(a, b + dt) \quad (\text{for all } t \in \mathbb{Q})$$

For brevity, let  $r = a/d$  and  $s = b/d$ . Dividing through by  $d$  gives, by elementary properties of the height,

$$1 \leq \eta(r, s + t)$$

Changing  $(r, s + t)$  by an element of  $\mathbb{Q}^\times$ , the idele  $r$  is a local unit at all finite primes of  $\mathbb{Q}$ . By right-multiplying by suitable

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard compact subgroups at finite primes, the idele  $r$  is 1 at all finite primes.

Given  $s \in \mathbb{A}$ , choose  $t \in \mathbb{Q}$  so that  $s + t$  is integral at all finite primes and  $|s + t|_\infty \leq \frac{1}{2}$ . With this  $t \in \mathbb{Q}$ , the height of  $(r, s + t)$  is

$$\eta(r, s + t) = \eta_\infty(r, s + t) = \sqrt{|r|_\infty^2 + |s + t|_\infty^2} \leq \sqrt{|r|_\infty^2 + \frac{1}{4}}$$

From  $1 \leq \eta(r, s + t)$

$$1 \leq |r|_\infty^2 + \frac{1}{4}$$

which gives

$$\frac{\sqrt{3}}{2} \leq |r|_\infty$$

Since  $r$  was a local unit at all finite primes,

$$\frac{\sqrt{3}}{2} \leq |r|$$

Since  $|r| = |a/d|$ ,

$$\frac{\sqrt{3}}{2} \leq \left| \frac{a}{d} \right|$$

This proves the theorem. ///

**[2.0.3] Remark:** This proof of Minkowski reduction uses the Euclidean-ness of  $\mathbb{Q}$ , and does not generalize simply to general situations. Rather, a relatively complicated argument *reduces* the general case to this. The general *conclusion* is analogous but the proof is different.

### 3. Imbeddings of arithmetic quotients

Let  $k$  be a number field. Let  $Q = \langle, \rangle$  be a non-degenerate quadratic form on a  $k$ -vectorspace  $V$ , and  $G = O(Q)$  the corresponding orthogonal group. We have the natural imbedding  $G \rightarrow GL(V)$ .

**[3.0.1] Proposition:** The inclusion  $G_k \rightarrow GL(V)_k$  induces an inclusion

$$G_k \backslash G_{\mathbb{A}} \rightarrow GL(V)_k \backslash GL(V)_{\mathbb{A}}$$

with closed image.

A general topological lemma is necessary.

**[3.0.2] Lemma:** Let  $X, Y$  be locally compact Hausdorff topological spaces. Further,  $X$  has a countable open cover  $\{U_i\}$  such that every  $U_i$  has compact closure. Let  $G$  be a group acting continuously on  $X$  and  $Y$ , *transitively* on  $X$ . Let  $f : X \rightarrow Y$  be a continuous *injective*  $G$ -set map whose image is a closed subset of  $Y$ . Then  $f$  is a *homeomorphism* of  $X$  to its image in  $Y$ .

*Proof:* This is a version of the Baire Category argument. Since  $f(X)$  is closed in  $Y$  the image  $f(X)$  is itself (with the subset topology) a locally compact Hausdorff space. Therefore, without loss of generality,  $f$  is *surjective*. Let  $C_i$  be the closure of  $U_i$ . The images  $f(C_i)$  of the  $C_i$  are compact, hence closed, by Hausdorff-ness. We claim that some  $f(C_i)$  must have non-empty interior. If not, we do the usual Baire argument: fix a non-empty open set  $V_1$  in  $Y$  with compact closure. Since  $f(C_1)$  contains no non-empty open set,  $V_1$  is not contained in  $f(C_1)$ , so there is a non-empty open set  $V_2$  whose closure is compact and whose closure is contained in  $V_1 - f(C_1)$ . Since  $f(C_2)$  cannot contain  $V_2$ , there is a non-empty open set  $V_3$  whose closure is compact and whose closure is contained in  $V_2 - f(C_2)$ . A descending chain of non-empty open sets is produced:

$$V_1 \supset \text{clos}(V_2) \supset V_2 \supset \text{clos}(V_3) \supset V_3 \supset \dots$$