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COHOMOLOGY OPERATIONS AND OBSTRUCTIONS TO EXTENDING CONTINUOUS FUNCTIONS

by

N. E. Steenrod

Colloquium Lectures

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§1. Introduction.

The class of problems known as extension problems is central to nearly all of topology. Many of the basic theorems of topology, and some of its most successful applications in other areas of mathematics are solutions of particular extension problems. The deepest results of this kind have been obtained by the method of algebraic topology. The essence of the method is a conversion of the geometric problem into an algebraic problem which is sufficiently complex to embody the essential features of the geometric problem, yet sufficiently simple to be solvable by standard algebraic methods. Many extension problems remain unsolved, and much of the current development of algebraic topology is inspired by the hope of finding a truly general solution.

To place my contribution to these developments in its proper setting, I will begin with a discussion of the extension problem, and the methods of finding solutions in special cases.

§2. The extension problem.

Let X and Y be topological spaces. Let A be a closed subset of X, and let $h: A \longrightarrow Y$ be a mapping, i.e. a continuous function from A to Y. A mapping $f: X \longrightarrow Y$ is called an extension of h if $f(x) = h(x)$ for each $x \in A$. The inclusion mapping $g: A \longrightarrow X$ is defined by $g(x) = x$ for $x \in A$.

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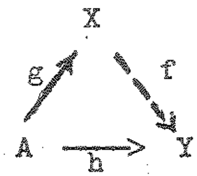
Then the condition that f be an extension can be restated: h is the composition fg of f and g.

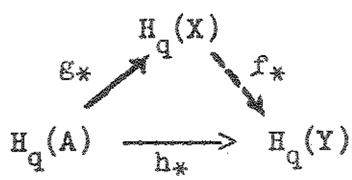


$$fg = h.$$

When X,Y,A and h are given, we have an extension problem: Does an extension f of h exist?

§3. Transforming geometric into algebraic problems.

The general method of attack on an extension problem is to apply homology theory to transform the problem into an algebraic problem. To the diagram of spaces and mappings we assign a diagram of groups and homomorphisms. Each space has a homology group H_q for each dimension q, and each mapping induces homomorphisms of the corresponding groups. Thus, for each q, we have an algebraic diagram



$$f_* g_* = h_*.$$

Given the three groups and the homomorphisms g_*, h_*, we can now ask the question: Does there exist a homomorphism ϕ such that $\phi g_* = h_*$? (It should be noted that g_* is not usually an inclusion, because a non-bounding cycle of A may bound in X). If an extension f exists, setting $\phi = f_*$ solves the algebraic problem because of the property $(fg)_* = f_* g_*$ of induced homomorphisms. Thus, the existence of a solution of the algebraic problem is a necessary condition for the existence of an extension. But it is not usually a sufficient condition. The reason for this is that much of the geometry has been lost in the transition to algebra.

It is a prime objective of research in algebraic topology to improve the algebraic machinery so as to give a sharper algebraic picture of the geometric problem. For example, in place of homology we may use cohomology. We obtain an analogous diagram

$$\begin{array}{ccccc} & & H^q(X) & & \\ & {\scriptstyle g^*}\swarrow & & \nwarrow{\scriptstyle f^*} & \\ H^q(A) & & \longleftarrow & & H^q(Y) \end{array} \qquad\qquad g^* f^* = h^*.$$

The chief difference is the reversal of the directions of the induced homomorphisms. If we consider cohomology solely as additive groups, they have no real advantage over homology groups. However, unlike homology, the cohomology groups of a space admit a ring structure: if $u \in H^p(Y)$ and $v \in H^q(Y)$, then they have a product, called the cup-product,

$$u \smile v \in H^{p+q}(Y).$$

This product is bilinear, and satisfies the commutative law $u \smile v = (-1)^{pq} v \smile u$. Furthermore a mapping $f\colon X \longrightarrow Y$ induces a ring homomorphism

$$f^*(u \smile v) = f^* u \smile f^* v.$$

Letting $H^*(Y) = \{H^q(Y),\ q = 0,1,\dots\}$ denote the resulting graded ring, the algebraic diagram becomes

$$\begin{array}{ccccc} & & H^*(X) & & \\ & {\scriptstyle g^*}\swarrow & & \nwarrow{\scriptstyle f^*} & \\ H^*(A) & & \underset{h^*}{\longleftarrow} & & H^*(Y) \end{array} \qquad\qquad g^* f^* = h^*,$$

and the algebraic problem is sharpened by the requirement that the solution ϕ of $g^*\phi = h^*$ must be a ring homomorphism.

This provides a considerable improvement in the algebraic picture of the geometric problem. However it is not the best that can be done. The cohomology groups possess not only a ring structure but also a more involved structure referred to as the system of cohomology operations. A cohomology operation T, relative to dimensions q and r, is a collection of functions $\{T_X\}$, one for each space X, such that

$$T_X: H^q(X) \longrightarrow H^r(X),$$

and, for each mapping $f: X \longrightarrow Y$,

$$f^* T_Y u = T_X f^* u \quad \text{for all} \quad u \in H^q(Y).$$

The simplest non-trivial operations are the squaring operations. For each dimension q and each integer $i \geqq 0$, there is a cohomology operation, called square-i,

$$Sq^i: H^q(X;Z_2) \longrightarrow H^{q+i}(X;Z_2).$$

Here the coefficient group Z_2 consists of the integers Z reduced modulo 2. Also for each prime $p > 2$, there are cohomology operations generalizing the squares called cyclic reduced p th powers. These are functions

$$\mathcal{P}^i: H^q(X; Z_p) \longrightarrow H^{q+2i(p-1)}(X; Z_p).$$

I will discuss these operations in detail later on. At the present time I wish only to emphasize the importance of cohomology operations to the study of the extension problem: In the derived algebraic problem using cohomology, the solution $\phi: H^*(Y) \longrightarrow H^*(X)$ of the algebraic problem $g^*\phi = h^*$ must be a ring homomorphism, and also must satisfy $\phi T_Y = T_X \phi$ for every cohomology operation T. Thus, by cramming as much structure as possible into cohomology theory, we endeavor to obtain the strongest possible necessary conditions for a solution of the extension problem.

The ultimate objective is to so refine the algebraic machinery that the derived algebraic problem is a faithful picture of the geometric problem. This has not yet been accomplished; but it appears to be within reach.

We turn now to a more detailed discussion of the ideas presented so far.

§4. Examples of extension problems.

Examples of solutions of extension problems are plentiful even in the most elementary aspects of topology. The Urysohn lemma is an example. In this case X is a normal space, $A = A_0 \cup A_1$ is the union of two disjoint closed subsets, Y is the interval $[0,1]$ of real numbers, and $h(A_0) = 0$, $h(A_1) = 1$. The conclusion of the lemma asserts that an extension always exists.

The Tietze extension theorem is another example. In this case X is normal, $Y = [0,1]$, and h is arbitrary. Again an extension always exists.

The study of the arcwise connectivity of a space Y is another example. In this case $X = [0,1]$, A consists of the two points 0 and 1, and $h(0) = y_0$, $h(1) = y_1$. An extension f of h is a path in Y from y_0 to y_1.

There is a special class of extension problems called retraction problems. If $A \subset X$, then a mapping $f: X \longrightarrow A$ is called a retraction if $f(x) = x$ for each $x \in A$. Given a space X and a closed subspace A, there is the problem of deciding whether or not such a retraction exists. By setting $Y = A$, and taking $h: A \longrightarrow Y$ to be the identity, it is seen that each retraction problem is an extension problem.

An important example from elementary algebraic topology is the following. Let E be the closed n-cell, i.e. the set $\Sigma_{i=1}^{n} x_i^2 \leqq 1$ in cartesian n-space, and let S be its boundary, i.e. the (n-1)-sphere $\Sigma_{i=1}^{n} x_i^2 = 1$. Then

The boundary S of the n-cell E is not a retract of E.

The proof of this for $n = 1$ is readily deduced from the fact that E is connected and S is not. For $n > 1$, the proof is not trivial, although the conclusion for $n = 2$ is intuitively appealing to anyone who has tightened a drum head, or stretched canvas tautly over a frame. The proof utilizes the general method of converting the problem into an algebraic one. We take homology groups in the dimension n-1, and obtain the diagram

$$\begin{array}{ccc} & H_{n-1}(E) & \\ {\scriptstyle g_*}\nearrow & & \searrow{\scriptstyle f_*} \\ H_{n-1}(S) & \xrightarrow[h_*]{} & H_{n-1}(S) \end{array}$$

The dimension n-1 is used since this gives the only non-trivial homology group of S. Using integer coefficients Z, we have $H_{n-1}(S) \approx Z$, and $H_{n-1}(E) = 0$. Now h = identity implies h_* = identity. This gives an impossibility: the identity homomorphism of Z cannot be factored into homomorphisms $Z \xrightarrow{g_*} 0 \xrightarrow{f_*} Z$. Therefore the retraction f of E into S does not exist.

It may be felt that a non-existence theorem is of little use. This is not the case. By a mild twist, a negative result can be given a positive form. In the case at hand, we obtain as a corollary the well-known Brouwer fixed-point theorem: Each mapping $g: E \longrightarrow E$ has at least one fixed point. For suppose to the contrary that there is a g with no fixed-point. As x and $g(x)$ are distinct points, they lie on a unique straight line, and x divides this line into two half lines. The half line not containing $g(x)$ meets S in a single point denoted by $f(x)$. The continuity of g implies that of f. In case

$x \in S$, it is clear that $f(x) = x$. So f is a retraction $E \longrightarrow S$. As this is impossible, a fixed point free g cannot exist.

§5. The use of the cohomology ring.

The next example is one in which the cohomology ring must be used to arrive at a decision. Let X denote the complex projective plane, i.e. the space of 3 homogeneous complex variables $[z_0, z_1, z_2]$ not all zero. It is a compact manifold of dimension 4. Let A be the complex projective line in X defined by the equation $z_2 = 0$. Topologically, A is a 2-sphere. In this case the conclusion is that A is not a retract of X.

Suppose that $f\colon X \longrightarrow A$ is a retraction so that fg = identity where $g\colon A \longrightarrow X$ is the inclusion. Passing to cohomology, we have the diagram

$$H^*(X) \underset{f^*}{\overset{g^*}{\rightleftarrows}} H^*(A), \qquad g^*f^* = \text{identity}.$$

When two groups are so related by homomorphisms, the left hand group splits into a direct sum:

$$H^*(X) = \text{Image of } f^* + \text{Kernel of } g^*$$

The abbreviated notation is

$$(5.1) \qquad H^*(X) = \text{Im } f^* + \text{Ker } g^*.$$

Furthermore g^* gives an isomorphism

$$(5.2) \qquad g^*\colon \text{Im } f^* \approx H^*(A).$$

If we include the ring structure, and use the fact that f^*, g^* are ring homomorphisms, then

(5.3) $\operatorname{Im} f^*$ is a subring, and $\operatorname{Ker} g^*$ is an ideal.

Turning to the example under consideration, we are given X, A and the inclusion g, and we can ask if $\operatorname{Ker} g^*$ is a direct summand. The cohomology of X is zero in dimensions > 4. In dimensions $\leqq 4$, the cohomology of X and A with integer coefficients Z is given by the table

	0	1	2	3	4
A	Z	0	Z	0	0
X	Z	0	Z	0	Z

Furthermore g^* is an isomorphism in the dimensions 0 and 2. It is seen then that the direct sum decomposition required by 5.1 does exist and, in fact, is unique. Namely, in the dimensions 0 and 2, $\operatorname{Ker} g^*$ is zero so that $\operatorname{Im} f^*$ is the whole group, and in the dimension 4, $\operatorname{Ker} g^*$ is the whole group and $\operatorname{Im} f^* = 0$.

However, on examining the ring structure, we find that the uniquely determined candidate for $\operatorname{Im} f^*$ is not a subring. For let u be a generator of $H^2(X)$ so that $u \in \operatorname{Im} f^*$. Since X is a manifold, the Poincaré duality theorem asserts that H^2 is self-dual under the cup product pairing to H^4. It follows that $u \smile u$ must generate $H^4(X)$. Therefore $u \smile u$ is not in $\operatorname{Im} f^*$; and therefore A is not a retract.

This example is intimately related to the mapping $h\colon S^3 \longrightarrow S^2$ of the 3-sphere into the 2-sphere studied first by H. Hopf [14]. In the space of two complex variables, let S^3 be the unit sphere $z_0\bar{z}_0 + z_1\bar{z}_1 = 1$, and E^4 the unit 4-cell $z_0\bar{z}_0 + z_1\bar{z}_1 \leqq 1$. Let S^2 be the space of two homogeneous complex variables $[z_0, z_1]$. Then h sends the point (z_0, z_1) of S^3 into $[z_0, z_1]$ in S^2. This is a very smooth mapping. The inverse images of points of S^2 give a fibration of S^3 into great circles. Hopf proved that h is not extendable to a mapping $E^4 \longrightarrow S^2$. (Notice that $(z_0, z_1) \longrightarrow [z_0, z_1]$ has a singularity at (0,0).) If we form a new space out of E^4 by collapsing its boundary S^3 into S^2 according to h, the resulting space is homeomorphic to the complex projective plane X, and S^2 corresponds to the complex projective line A. Since A is not a retract of X, it follows that h cannot be extended over E^4.

§6. The use of the squaring operations.

The next example is a retraction problem for which the cohomology ring does not provide an answer; but the squaring operations do give an answer. Let P^5 denote the real projective space of dimension 5 (6 homogeneous real variables). Let $P^4 \supset P^3 \supset P^2$ be projective subspaces of the indicated dimensions. Let X be the space obtained from P^5 by collapsing P^2 to a point, and let $A \subset X$ be the image of P^4 under the collapsing map : $P^5 \longrightarrow X$. Again the assertion is that A is not a retract of X.

We tackle this problem in the same manner as the preceding one, and begin by asking whether Ker g^* is a direct summand of $H^*(X)$. Knowing the cohomology of P^5, one readily deduces that of X and A. With Z_2 as coefficients, the cohomology is given by the following table

	0	1	2	3	4	5
A	Z_2	0	0	Z_2	Z_2	0
X	Z_2	0	0	Z_2	Z_2	Z_2

Furthermore, g^* is an isomorphism in dimensions < 5. Therefore there is a direct sum splitting as in 5.1 and it is unique: Im f^* must be the whole group in dimensions < 5, and it is zero in the dimension 5.

In this case the candidate for $\mathrm{Im}\, f^*$ is obviously a subring. The reason is that the cup product of elements of $\dim \geqq 3$ has $\dim \geqq 6$, and is therefore zero. Thus, insofar as the cohomology ring is concerned, A could be a retract of X. To show that it is not a retract, we must use the cohomology operation

$$Sq^2 : H^3(X; Z_2) \longrightarrow H^5(X; Z_2).$$

If u is the non-zero element of H^3, a suitable calculation shows that $Sq^2 u$ is the non-zero element of H^5. Now the unique candidate for $\mathrm{Im}\, f^*$ contains u and is zero in dimension 5; hence it is not closed under Sq^2. But it would have to be closed if a retraction f existed because $f^* Sq^2 = Sq^2 f^*$. Therefore a retraction does not exist.

This result has a good application in differential geometry. It is well known that a differentiable manifold has a continuous field F of non-zero tangent vectors if and only if its Euler number is zero. This implies that the n-sphere S^n has a tangent field F if and only if n is odd. S^3 in fact has 3 fields which are independent at each point because it is a group manifold (unit quaternions). The question arises as to the maximum number of fields tangent to S^5 which are independent at each point. The answer is 1. For, by a direct construction, two independent fields can be made to yield a retraction of X into A (see [28]).

The same method can be used to prove a more general result [28]. If n is a positive integer, and 2^k is the largest power of 2 dividing $n+1$, then any set of 2^k vector fields tangent to S^n are dependent at some point. This result is the best possible for $n < 15$.

§7. Homotopies.

Having demonstrated the need of finer and finer algebraic tools, it is natural to ask if there is an end to the process. The answer is that there is real hope of achieving a complete solution. To exhibit the basis for my hope, I must delve more deeply into the geometric aspects of the extension problem. For this, the concept of homotopy is vital. Let h be a mapping $A \longrightarrow Y$, and let $I = [0,1]$ be the unit interval, then a mapping $H: A \times I \longrightarrow Y$ is called a homotopy of h if $H(x,0) = h(x)$ for $x \in A$. Setting $h'(x) = H(x,1)$, H is called a homotopy of h into h' and we write $h \simeq h'$ (h is homotopic to h'). This is an equivalence relation, and the set of maps homotopic to h is called the homotopy class of h. The set of homotopy classes of mappings $A \longrightarrow Y$ is denoted by $\mathrm{Map}(X, Y)$.

A basic result, due to Borsuk, is the

Homotopy Extension Theorem. If $f: X \longrightarrow Y$, A is closed in X, and $h = f|A$. Then any homotopy H of h may be extended to a homotopy of f. Precisely, the mapping G of the subset $X \times 0 \cup A \times I$ of $X \times I$ into Y, given by $G(x,0) = f(x)$ for $x \in X$ and $G(x,t) = H(x,t)$ for $x \in A$, $t \in I$, may be extended to a mapping $F: X \times I \longrightarrow Y$.

The intuitive idea of the theorem is that if we grab hold of the image of A and pull it along, then the image of X will come sliding after.

The theorem is not true in the generality stated; some restriction on X, A or Y is necessary. It suffices for example if Y is triangulable or if X and A are triangulable. It also suffices to impose the condition of being an absolute neighborhood retract on Y or on X and A. In the future we assume some such restriction without further mention.

Notice that the theorem asserts the extendability of certain kinds of mappings. This solution of a special extension problem is of the utmost importance for the general problem because of the following

Corollary. The extendability of $h: A \longrightarrow Y$ to a mapping $f: X \longrightarrow Y$ depends only on the homotopy class of h: If h is extendable and $h \simeq h'$, then h' is extendable.

It is only necessary to extend the homotopy to $F: X \times I \longrightarrow Y$ and set $f'(x) = F(x,1)$.

One advantage this gives us is that, in any particular extension problem, we may vary h by a homotopy and obtain a simpler but equivalent problem. For example, suppose it were known that h is homotopic to a constant mapping h' (i.e. $h'(A)$ is a single point). Since such an h' is obviously extendable, so is h.

The result also enables us to rephrase the extension problem in an apparently weaker form: Does there exist an f such that $fg \simeq h$? Given such an f, we have that $f|A$ is obviously extendable, and $f|A \simeq h$, and so h is extendable.

Having freed one aspect of the extension problem (replacing $fg = h$ by $fg \simeq h$), it is natural to consider freeing other parts of unnecessary restrictions. The condition that g be the inclusion mapping $A \subset X$ is no longer an essential feature. Let X, A, Y be any three spaces and let $h: A \longrightarrow Y$ and $g: A \longrightarrow X$ be mappings. Does there exist a mapping $f: X \longrightarrow Y$ such that $fg \simeq h$? This problem is called the "left-factorization" problem. The class of these problems includes the extension problems and many more. Broadening thus the class of problems does not increase the difficulties because of the following result.

<u>Each left-factorization problem is equivalent to some retraction problem.</u>

To see this, we start with a left-factorization problem as above, and construct a space Z as follows. In the union of X, $A \times I$ and Y, identify each point $(a,0)$ with $g(a)$ in x, and identify each point $(a,1)$ with $h(a)$ in Y. The resulting space Z contains X and Y and a homotopy of g into h. It follows quickly that Y is a retract of Z if and only if there exists a mapping $f: X \longrightarrow Y$ such that $fg \simeq h$.

Thus the broadest type of problem is equivalent to the narrowest type.

It is easily shown that a left-factorization problem depends only on the homotopy classes of g and h. Even more it depends only on the <u>homotopy types</u> of the three spaces involved. Two spaces X, X' have the same homotopy type (are homotopically equivalent) if there exist mappings $\phi: X \longrightarrow X'$ and $\phi': X' \longrightarrow X$ such that $\phi\phi' \simeq$ identity of X' and $\phi'\phi \simeq$ identity of X. We may substitute X' for X in any problem if we set $g' = \phi g$. Analogous substitutions can be made for A and Y.

An advantage of this flexibility is that any particular problem can often be greatly simplified by homotopic alterations of the spaces and mappings involved.

More important however is the light which it casts on the class of all problems. If we consider only those spaces admitting finite triangulations, then there are only a countable number of homotopy types of spaces, and for any two spaces there are only a countable number of homotopy classes of mappings. This statement can be proved by the use of the well-known simplicial approximation theorem. It is a consequence that <u>there are only a countable number of extension problems</u>. This in itself makes it reasonable to hope for effective methods of solving any extension problem.

To substantiate this hope, consider the notion of the induced homomorphism

f^* of cohomology assigned to a mapping $f\colon X \longrightarrow Y$. A well-known property is that homotopic maps induce the same homomorphism. Hence we have a function

$$R_{XY}\colon \mathrm{Map}(X,Y) \longrightarrow \mathrm{Hom}(H^*(Y),H^*(X))$$

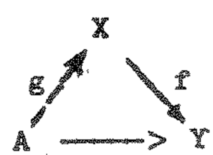
defined by $R_{XY}(f) = f^*$. By <u>Hom</u> we mean all functions preserving whatever algebraic structure we are able to put into the cohomology theory of spaces. Suppose we have an extension problem with spaces X, A, Y such that R_{XY} is <u>onto</u>, and R_{AY} is 1-1 into. Suppose moreover that the algebraic problem $g^*\phi = h^*$ has a solution ϕ. Since R_{XY} is onto, there exists an $f\colon X \longrightarrow Y$ such that $f^* = \phi$. Then $(fg)^* = h^*$. Since R_{AY} is 1-1 into, this implies $fg \simeq h$. Hence the solvability of the algebriac problem is both necessary and sufficient for solving the geometric problem.

Thus we would have a complete hold on the extension problem if we knew that R_{XY} is 1-1 onto for all triangulable spaces X, Y. This is true for some spaces and false for others. For example, let $X = S^3$ and $Y = S^2$; then $\mathrm{Hom}(H^*(S^2),H^*(S^3)) = 0$, and $\mathrm{Map}(S^3,S^2) = \pi_3(S^2)$ is infinite. However our point of view above has been too narrow in specifying the range of R_{XY}. Some more intricate algebraic gadget should do the trick. The possibilities are many. For example $R_{XY}(f)$ could be taken to be the cohomology sequence associated with the mapping cylinder of f.

The finding of a suitable 1-1 mapping R_{XY} of $\mathrm{Map}(X,Y)$ into a computable algebraic object is called the <u>homotopy classification problem</u>. Solving it completely will solve the extension problem completely. Why should we be hopeful of solving this? First, $\mathrm{Map}(X,Y)$ is a countable set, and is therefore suitable for algebraization. Secondly, in many special cases (as will be shown) we have obtained solutions. Thirdly, we have available now a variety of functions R_{XY} which taken together may provide the complete solution.

§8. Lifting problems.

There is a class of problems called lifting problems which are dual in a certain sense to extension problems. In a lifting problem, we are given a fibre bundle X over a base space Y with projection $f\colon X \longrightarrow Y$. This means that each $y \in Y$ has a neighborhood V such that $f^{-1}V$ is representable as a product space $V \times F$ for some fixed space F called the fibre. Furthermore, f restricted to $f^{-1}V$ is the projection $V \times F \longrightarrow V$. In the lifting problem, we are also given a space A and a mapping $h\colon A \longrightarrow Y$; and the problem is to decide whether there exists a mapping $g\colon A \longrightarrow X$ such that $fg = h$.

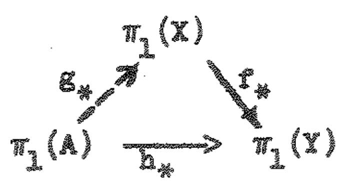


$fg = h.$

The condition that $X \xrightarrow{f} Y$ is a bundle is dual to the condition of an extension problem that $A \xrightarrow{g} X$ is an inclusion mapping.

An elementary example of a lifting problem and its solution is the

Monodromy theorem. If X is a covering space of Y with projection f, then a mapping $h\colon A \longrightarrow Y$ can be lifted to $g\colon A \longrightarrow X$ if and only if the algebraic problem posed by the fundamental groups has a solution:



$f_* g_* = h_*$.

It should be recalled that, since X covers Y, f_* imbeds $\pi_1(X)$ isomorphically into $\pi_1(Y)$. Also, since base points are not specified, the images